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THE

CAMBRIDGE AND DUBLIN

MATHEMATICAL JOURNAL.

ON THE INDETERMINATE CURVATURE OF SURFACES.

By William Walton.

If, through an indefinitely small arc ds at any point x, y, z, of a surface $F = 0$, a normal plane be drawn cutting the surface, the ordinary formula for the determination of the radius of curvature ρ of the section at the point is

$$\rho^2 = \frac{U^2 + V^2 + W^2}{(l^2u + m^2v + n^2w + 2mnu' + 2nlv' + 2lmw')^2},$$

where l, m, n, are the direction-cosines of ds, and where

$$U = \frac{dF}{dx}, \quad V = \frac{dF}{dy}, \quad W = \frac{dF}{dz},$$

$$u = \frac{d^2F}{dx^2}, \quad v = \frac{d^2F}{dy^2}, \quad w = \frac{d^2F}{dz^2},$$

$$u' = \frac{d^2F}{dy\,dz}, \quad v' = \frac{d^2F}{dz\,dx}, \quad w' = \frac{d^2F}{dx\,dy};$$

l, m, n, being connected by the condition

$$lU + mV + nW = 0.$$

Suppose, however, that U, V, W, are simultaneously zero at any point; then the second differential of the equation to the surface gives us

$$Ud^2x + Vd^2y + Wd^2z + udx^2 + vdy^2 + wdz^2 + 2u'dydz + 2v'dzdx + 2w'dxdy = 0,$$

and therefore

$$l^2u + m^2v + n^2w + 2mnu' + 2nlv' + 2lmw' = 0.$$

Thus the expression for ρ^2 assumes the indeterminate form $\frac{0}{0}$ and becomes inapplicable. This indeterminateness belongs

to a singular point or to any point on a singular line of a surface. My object in this paper is to supply this defect in the theory of curvature by the investigation of a formula applicable to such anomalous cases.

The tangent plane at any point $x+\delta x$, $y+\delta y$, $z+\delta z$, on the surface, is represented by the equation

$$(x_1 - x - \delta x)(U+\delta U) + (y_1 - y - \delta y)(V+\delta V) + (z_1 - z - \delta z)(W+\delta W) = 0.$$

If x, y, z, be a point at which $U=0$, $V=0$, $W=0$; then

$$(x_1 - x - \delta x)\,\delta U + (y_1 - y - \delta y)\,\delta V + (z_1 - z - \delta z)\,\delta W = 0,$$

or, in the limit,

$$(x_1 - x)\frac{dU}{ds} + (y_1 - y)\frac{dV}{ds} + (z_1 - z)\frac{dW}{ds} = 0.$$

The equation to the normal plane through ds, is

$$L(x_1 - x) + M(y_1 - y) + N(z_1 - z) = 0 \ldots\ldots (1),$$

where L, M, N, are subject to the conditions

$$L\frac{dU}{ds} + M\frac{dV}{ds} + N\frac{dW}{ds} = 0 \ldots\ldots\ldots (2),$$

$$Ll + Mm + Nn = 0 \ldots\ldots\ldots (3),$$

l, m, n, being connected together by the equation

$$l^2 + m^2 + n^2 = 1 \ldots\ldots\ldots\ldots (4).$$

Eliminating L, M, N, between (1), (2), (3), we get, for the equation to the normal plane,

$$(x_1 - x)\left(n\frac{dV}{ds} - m\frac{dW}{ds}\right) + (y_1 - y)\left(l\frac{dW}{ds} - n\frac{dU}{ds}\right) + (z_1 - z)\left(m\frac{dU}{ds} - l\frac{dV}{ds}\right) = 0 \ldots\ldots\ldots (5).$$

At the intersection of the surface

$$F = 0 \ldots\ldots\ldots\ldots\ldots (6),$$

with the plane (5), at the point x, y, z, if x', y', z', be the centre of the osculating circle, and λ, μ, ν, be taken to represent $\frac{dl}{ds}$, $\frac{dm}{ds}$, $\frac{dn}{ds}$, respectively, we know that

$$\left.\begin{aligned} x' - x = \lambda\rho^2, \quad y' - y = \mu\rho^2, \quad z' - z = \nu\rho^2, \\ \rho^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2 \end{aligned}\right\} \ldots\ldots (7).$$

Differentiating the equation (4), we have

$$l\lambda + m\mu + n\nu = 0 \quad \ldots\ldots\ldots\ldots \quad (8).$$

Differentiating the equation (6) twice, we get

$$l\frac{dU}{ds} + m\frac{dV}{ds} + n\frac{dW}{ds} + \lambda U + \mu V + \nu W = 0,$$

which becomes, at the point x, y, z,

$$l\frac{dU}{ds} + m\frac{dV}{ds} + n\frac{dW}{ds} = 0 \quad \ldots\ldots\ldots \quad (9),$$

and, by a third differentiation,

$$2\lambda\frac{dU}{ds} + 2\mu\frac{dV}{ds} + 2\nu\frac{dW}{ds} + l\frac{d^2U}{ds^2} + m\frac{d^2V}{ds^2} + n\frac{d^2W}{ds^2} = 0$$

$$\ldots\ldots (10).$$

Also, differentiating the equation (5) twice with regard to x_1, y_1, z_1, and then making x_1, y_1, z_1, coincide with x, y, z, we obtain

$$\lambda\left(n\frac{dV}{ds} - m\frac{dW}{ds}\right) + \mu\left(l\frac{dW}{ds} - n\frac{dU}{ds}\right)$$

$$+ \nu\left(m\frac{dU}{ds} - l\frac{dV}{ds}\right) = 0 \quad \ldots\ldots\ldots \quad (11).$$

From (8) and (11) we have

$$\mu\left\{(l^2 + m^2)\frac{dW}{ds} - n\left(l\frac{dU}{ds} + m\frac{dV}{ds}\right)\right\}$$

$$= \nu\left\{(l^2 + n^2)\frac{dV}{ds} - m\left(l\frac{dU}{ds} + n\frac{dW}{ds}\right)\right\},$$

and therefore, by (9),

$$\mu\frac{dW}{ds} = \nu\frac{dV}{ds},$$

and, by symmetry,

$$\frac{1}{\lambda}\frac{dU}{ds} = \frac{1}{\mu}\frac{dV}{ds} = \frac{1}{\nu}\frac{dW}{ds} \quad \ldots\ldots\ldots \quad (12).$$

Now $$\frac{dU}{ds} = lu + mw' + nv',$$

$$\frac{d^2U}{ds^2} = l\frac{du}{ds} + m\frac{dw'}{ds} + n\frac{dv'}{ds} + \lambda u + \mu w' + \nu v',$$

and, similarly,

$$\frac{d^2V}{ds^2} = m\frac{dv}{ds} + n\frac{du'}{ds} + l\frac{dw'}{ds} + \mu v + \nu u' + \lambda w',$$

and $$\frac{d^2W}{ds^2} = n\frac{dw}{ds} + l\frac{dv'}{ds} + m\frac{du'}{ds} + \nu w + \lambda v' + \mu u'.$$

Hence

$$\begin{aligned} & l\frac{d^2U}{ds^2} + m\frac{d^2V}{ds^2} + n\frac{d^2W}{ds^2} \\ &= l^2\frac{du}{ds} + m^2\frac{dv}{ds} + n^2\frac{dw}{ds} \\ &\quad + 2mn\frac{du'}{ds} + 2nl\frac{dv'}{ds} + 2lm\frac{dw'}{ds} \\ &\quad + \lambda(lu + mw' + nv') \\ &\quad + \mu(mv + nu' + lw') \\ &\quad + \nu(nw + lv' + mu') \\ &= l^2\frac{du}{ds} + m^2\frac{dv}{ds} + n^2\frac{dw}{ds} \\ &\quad + 2mn\frac{du'}{ds} + 2nl\frac{dv'}{ds} + 2lm\frac{dw'}{ds} \\ &\quad + \lambda\frac{dU}{ds} + \mu\frac{dV}{ds} + \nu\frac{dW}{ds}. \end{aligned}$$

Hence the equation (10) becomes

$$\begin{aligned} & 3\lambda\frac{dU}{ds} + 3\mu\frac{dV}{ds} + 3\nu\frac{dW}{ds} \\ & + l^2\frac{du}{ds} + m^2\frac{dv}{ds} + n^2\frac{dw}{ds} \\ & + 2mn\frac{du'}{ds} + 2nl\frac{dv'}{ds} + 2lm\frac{dw'}{ds} = 0 \quad \ldots\ldots (13). \end{aligned}$$

From (12) and (13) there is

$$\frac{3\left(\frac{dU^2}{ds^2} + \frac{dV^2}{ds^2} + \frac{dW^2}{ds^2}\right)}{l^2\frac{du}{ds} + m^2\frac{dv}{ds} + n^2\frac{dw}{ds} + 2mn\frac{du'}{ds} + 2nl\frac{dv'}{ds} + 2lm\frac{dw'}{ds}} = -\frac{1}{\lambda}\frac{dU}{ds} = -\frac{1}{\mu}\frac{dV}{ds} = -\frac{1}{\nu}\frac{dW}{ds},$$

and therefore, by (7),

$$\frac{9}{\rho^2} = \frac{\left(l^2\frac{du}{ds} + m^2\frac{dv}{ds} + n^2\frac{dw}{ds} + 2mn\frac{du'}{ds} + 2nl\frac{dv'}{ds} + 2lm\frac{dw'}{ds}\right)^2}{\frac{dU^2}{ds^2} + \frac{dV^2}{ds^2} + \frac{dW^2}{ds^2}}.$$

If we substitute for $\frac{du}{ds}$ its value

$$l\frac{du}{dx} + m\frac{du}{dy} + n\frac{du}{dz},$$

and make like substitutions for $\frac{dv}{ds}, \frac{dw}{ds}, \ldots$, the square root of the numerator of this fraction will assume the form

$$\begin{aligned} &l^2\left\{l\frac{du}{dx} + m\frac{du}{dy} + n\frac{du}{dz} + 2m\frac{dw'}{dx} + 2n\frac{dv'}{dx}\right\} \\ &+ m^2\left\{m\frac{dv}{dy} + n\frac{dv}{dz} + l\frac{dv}{dx} + 2n\frac{du'}{dy} + 2l\frac{dw'}{dy}\right\} \\ &+ n^2\left\{n\frac{dw}{dz} + l\frac{dw}{dx} + m\frac{dw}{dy} + 2l\frac{dv'}{dz} + 2m\frac{du'}{dz}\right\} \\ &+ 2lmn\left\{\frac{du'}{dx} + \frac{dv'}{dy} + \frac{dw'}{dz}\right\}, \end{aligned}$$

the denominator, when expressed in terms of partial differential coefficients, being equal to

$$(lu + mw' + nv')^2 + (mv + nu' + lw')^2 + (nw + lv' + mu')^2.$$

Also the equation (9) may be written

$$l^2u + m^2v + n^2w + 2mnu' + 2nlv' + 2lmw' = 0 \ldots (14).$$

Ex. To find the radius of the osculating circle of any normal section of the Cono-cuneus of Wallis at any point of its singular line.

The equation to the surface is

$$F = a^2y^2 + x^2(z^2 - c^2) = 0.$$

We have therefore

$$\begin{aligned} &U = 2x(z^2 - c^2), \quad &&V = 2a^2y, \quad &&W = 2x^2z, \\ &u = 2(z^2 - c^2), \quad &&v = 2a^2, \quad &&w = 2x^2, \\ &u' = 0, \quad &&v' = 4xz, \quad &&w' = 0, \end{aligned}$$

$$\frac{du}{dx} = 0, \quad \frac{du}{dy} = 0, \quad \frac{du}{dz} = 4z,$$

$$\frac{dv}{dx} = 0, \quad \frac{dv}{dy} = 0, \quad \frac{dv}{dz} = 0,$$

$$\frac{dw}{dx} = 4x, \quad \frac{dw}{dy} = 0, \quad \frac{dw}{dz} = 0,$$

$$\frac{du'}{dx} = 0, \quad \frac{du'}{dy} = 0, \quad \frac{du'}{dz} = 0,$$

$$\frac{dv'}{dx} = 4z, \quad \frac{dv'}{dy} = 0, \quad \frac{dv'}{dz} = 4x,$$

$$\frac{dw'}{dx} = 0, \quad \frac{dw'}{dy} = 0, \quad \frac{dw'}{dz} = 0.$$

The quantities U, V, W, all vanish if $x = 0$, $y = 0$, whatever be the value of z; the axis of z being therefore a singular line.

Putting $x = 0$, $y = 0$, in these expressions for the partial differential coefficients of F, we have

$$\frac{9}{\rho^2} = \frac{(12l^2nz)^2}{4l^2(z^2 - c^2)^2 + 4m^2a^4},$$

$$\frac{1}{\rho^2} = 4z^2 \frac{l^4n^2}{l^2(z^2 - c^2)^2 + m^2a^4} \quad \ldots\ldots\ldots (1)'.$$

Also the equation (14) becomes

$$l^2(z^2 - c^2) + m^2a^2 = 0 \quad \ldots\ldots\ldots\ldots (2)'.$$

If from the equation

$$l^2 + m^2 + n^2 = 1,$$

combined with (2)′, we obtain l^2, m^2, in terms of n^2, and substitute in (1)′, we shall eventually get

$$\frac{1}{\rho^2} = n^2(1 - n^2) \frac{4a^2z^2}{(c^2 - z^2)(a^2 + c^2 - z^2)^2}.$$

It is easily seen, from this result, that, for any proposed value of z, the curvature will be a maximum when $n = \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$, and a minimum when $n = 0$.

Thus the maximum value of the radius of curvature is ∞, and each of its two minimum values is equal to the square root of

$$\frac{(c^2 - z^2)(a^2 + c^2 - z^2)^2}{a^2 z^2},$$

Cambridge, July 21, 1851.

ON THE CIRCULAR SECTIONS OF SURFACES OF THE SECOND ORDER.

By William Walton.

In a paper, in the fourth volume of the *Cambridge Mathematical Journal*, I have applied the method of Indeterminate Maxima and Minima, among other questions, to the determination of the positions of the circular sections through the centre of an ellipsoid referred to its axes. The same method may be conveniently adopted in investigating the positions of such sections and of their centric loci in surfaces of the second order referred to any system of rectangular axes. The results at which I have arrived in this paper coincide with those given by Mr. Frost and Professor Thomson in the first volume of the second series of the *Mathematical Journal*.

The equation to any surface of the second order is

$$\phi(x, y, z) = ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy + 2a''x + 2b''y + 2c''z + f = 0 \ldots\ldots\ldots (1).$$

Let x_1, y_1, z_1, be the coordinates of the centre of any plane section, of which l, m, n, are the direction-cosines and r a radius vector originating in the centre of the section. Then, l', m', n', being the direction-cosines of r in any position, and x, y, z, the coordinates of its extremity,

$$x = x_1 + l'r, \quad y = y_1 + m'r, \quad z = z_1 + n'r.$$

If we substitute these values of x, y, z, in (1), we shall have a quadratic in r. The coefficient of r in this quadratic must be zero, the origin of the radii vectores being the centre of the section. Hence we see that, ϕ being written for $\phi(x_1, y_1, z_1)$,

$$r^2(al'^2 + bm'^2 + cn'^2 + 2a'm'n' + 2b'n'l' + 2c'l'm') + \phi = 0 \ldots (2),$$

and

$$l' \frac{d\phi}{dx_1} + m' \frac{d\phi}{dy_1} + n' \frac{d\phi}{dz_1} = 0. \ldots\ldots\ldots (3).$$

The quantities l', m', n', are subject to the two relations

$$ll' + mm' + nn' = 0 \ldots\ldots\ldots\ldots (4),$$

$$l'^2 + m'^2 + n'^2 = 1 \ldots\ldots\ldots\ldots\ldots (5).$$

Now, generally, r^2 will have a maximum and a minimum value by virtue of the equations (2), (4), (5), for any system of values of l, m, n. Our object is to proceed with the investigation for finding the corresponding values of l', m', n', and then to assign such values to l, m, n, as shall render l', m', n', indeterminate.

Before proceeding with this investigation, we may observe that, when l', m', n', are indeterminate, the two equations (3) and (4) must be identical, and that accordingly

$$\frac{1}{l}\frac{d\phi}{dx_1} = \frac{1}{m}\frac{d\phi}{dy_1} = \frac{1}{n}\frac{d\phi}{dz_1} \ldots\ldots\ldots (6),$$

which are the equations to the locus of the centres corresponding to a system of values of l, m, n, determined in the manner above described.

If we put

$$P = al'^2 + bm'^2 + cn'^2 + 2a'm'n' + 2b'n'l' + 2c'l'm' \ldots (7),$$

it is plain that r^2 will have a maximum or minimum value whenever P has a minimum or a maximum.

Now, from (4), we see that

$$2m'n' = \frac{l'^2 - m'^2 - n'^2}{mn}, \quad 2n'l' = \frac{m'^2 - n'^2 - l'^2}{nl},$$

$$2l'm' = \frac{n'^2 - l'^2 - m'^2}{lm}.$$

Substituting these values of $2m'n'$, $2n'l'$, $2l'm'$, in (7), we have

$$P = Hl'^2 + Km'^2 + Ln'^2 \ldots\ldots\ldots\ldots (8),$$

where

$$\left.\begin{aligned} H &= a + \frac{l}{mn}(la' - mb' - nc') \\ K &= b + \frac{m}{nl}(mb' - nc' - la') \\ L &= c + \frac{n}{lm}(nc' - la' - mb') \end{aligned}\right\} \ldots\ldots\ldots (9).$$

Differentiating (4), (5), (8), considering l', m', n', variable, and putting $dP = 0$, we have, by the aid of indeterminate multipliers λ, μ,

$$\lambda l + (\mu + H)l' = 0, \quad \lambda m + (\mu + K)m' = 0, \quad \lambda n + (\mu + L)n' = 0.$$

Multiplying these equations in order by l', m', n', adding, and attending to (4), (5), (8), we have

$$\mu + P = 0,$$

and therefore

$$l' = \frac{\lambda l}{P - H}, \quad m' = \frac{\lambda m}{P - K}, \quad n' = \frac{\lambda n}{P - L}.$$

Assuming that l', m', n', are indeterminate, we have

$$\lambda = 0, \quad P = H = K = L.$$

Thus the equations (9) become, a, β, γ, representing $a - P$, $b - P$, $c - P$, respectively,

$$amn + l^2a' - lmb' - lnc' = 0,$$

$$\beta nl + m^2b' - mnc' - mla' = 0,$$

$$\gamma lm + n^2c' - nla' - nmb' = 0.$$

Multiplying the first of these three equations by n, the third by l, adding, and dividing by m, we get

$$\left.\begin{aligned} & an^2 + \gamma l^2 - 2lnb' = 0. \\ \text{Similarly} \quad & \beta l^2 + am^2 - 2mlc' = 0, \\ \text{and} \quad & \gamma m^2 + \beta n^2 - 2nma' = 0. \end{aligned}\right\} \ldots\ldots\ldots (11).$$

Eliminating mn, m^2, n^2, from the first and last three of the last six equations, we obtain

$$\left.\begin{aligned} & (a'^2 - \beta\gamma)\, l + (c'\gamma - a'b')\, m + (b'\beta - c'a')\, n = 0. \\ \text{Similarly we may shew that} & \\ & (b'^2 - \gamma a)\, m + (a'a - b'c')\, n + (c'\gamma - a'b')\, l = 0, \\ & (c'^2 - a\beta)\, n + (b'\beta - c'a')\, l + (a'a - b'c')\, m = 0. \end{aligned}\right\} \ldots(12).$$

Eliminating n between the first two of the equations (12), we get

$$(a'l - b'm)\, S = 0,$$

where $$S = a'^2a + b'^2\beta + c'^2\gamma - a\beta\gamma - 2a'b'c'.$$

Similarly there is also

$$(b'm - c'n)\, S = 0,$$

$$(c'n - a'l)\, S = 0.$$

Thus we see that either $S = 0 \ldots\ldots\ldots\ldots\ldots\ldots (13)$,

or $$a'l = b'm = c'n \ldots\ldots\ldots\ldots\ldots (14).$$

If $S = 0$, the equations (12) must be identical, and therefore

$$(b'\beta - c'a')(c'\gamma - a'b') = (a'a - b'c')(a'^2 - \beta\gamma).$$

This relation reduces the first of the equations (12) to the form

$$\frac{l}{a'\alpha - b'c'} + \frac{m}{b'\beta - c'a'} + \frac{n}{c'\gamma - a'b'} = 0 \ldots\ldots (15).$$

This equation, combined with any one of the equations (11) and the equation (13), will determine the values of l, m, n.

If the equations (14) be true, then, by (9),

$$\alpha = \frac{b'c'}{a'}, \quad \beta = \frac{c'a'}{b'}, \quad \gamma = \frac{a'b'}{c'} \ldots\ldots (16),$$

and the equation (13) is still satisfied, the equation (15) assuming a nugatory form. In this case the equations to the centric loci become

$$a' \frac{d\phi}{dx_1} = b' \frac{d\phi}{dy_1} = c' \frac{d\phi}{dz_1},$$

or, writing these equations in full, and attending to the relations (16),

$$a' (Px_1 + a'') = b' (Py_1 + b'') = c' (Pz_1 + c'').$$

These equations shew that the circular sections, of which the direction-cosines are given by (14), are all perpendicular to their centric locus. Thus the surface is one of revolution.

Cor. Suppose that $a = b$ and $c' = 0$. Then also $\alpha = \beta$. The equations (11) then become

$$an^2 + \gamma l^2 - 2lnb' = 0, \quad a(l^2 + m^2) = 0, \quad \gamma m^2 + an^2 - 2nma' = 0.$$

These equations are satisfied by

$$l = 0, \quad m = 0, \quad a = 0.$$

Thus one series of circular sections are parallel to the plane of xy.

August 23, 1851.

DEMONSTRATION OF BRIANCHON'S THEOREM, AND OF AN ANALOGOUS PROPERTY IN SPACE.

By Thomas Weddle.

Let $ABCDEF$ be a hexagon circumscribed about a conic, and

$$t = 0, \quad u = 0, \quad v = 0,$$

the equations to the straight lines CE, AE, and AC, respectively. Supposing t, u, and v to have been multiplied

by the proper constants, we may denote the equation to the conic by

$$t^2 + u^2 + v^2 + 2\lambda uv + 2\mu tv + 2\nu tu = 0 \ldots\ldots (1).$$

This may be put under the form

$$(t + \mu v + \nu u)^2 + (1 - \nu^2)\, u^2 + (1 - \mu^2)\, v^2 + 2(\lambda - \mu\nu)\, uv = 0,$$

from which we see that

$$(1 - \nu^2)\, u^2 + (1 - \mu^2)\, v^2 + 2(\lambda - \mu\nu)\, uv = 0 \ldots . (2),$$

is the equation to the two tangents drawn from A; that is, of the two straight lines AB and AF. Now by assigning a proper value to l, the expression

$$-\frac{1}{l}\{lv - (1 - \nu^2)\, u\}\ \{lu - (1 - \mu^2)\, v\}$$

can be rendered identical with the left-hand member of (2); hence the equations to AB and AF are $lv = (1 - \nu^2)\, u$ and $lu = (1 - \mu^2)\, v$.

In a similar manner the equations to the other sides of the hexagon may be got; and, collecting the whole, we have the

$$\left.\begin{array}{rr} \text{equation to } CD, & mt = (1 - \lambda^2)\, v \\ DE, & nt = (1 - \lambda^2)\, u \\ AF, & lu = (1 - \mu^2)\, v \\ EF, & nu = (1 - \mu^2)\, t \\ AB, & lv = (1 - \nu^2)\, u \\ BC, & mv = (1 - \nu^2)\, t \end{array}\right\} \ldots\ldots\ldots (3).$$

Eliminate t from the first and second of these equations, u from the third and fourth, and v from the fifth and sixth; and we get the

$$\left.\begin{array}{rr} \text{equation of } AD, & mu = nv \\ CF, & lt = nv \\ BE, & lt = mu \end{array}\right\} \ldots\ldots\ldots (4).$$

Hence the diagonals AD, CF, and BE intersect in a point whose equations are

$$lt = mu = nv \ldots\ldots\ldots\ldots (5);$$

and thus Brianchon's theorem is established.

The analogue referred to in the title is that due to M. Chasles, various forms of which are given in my third memoir on the "Theorems in Space analogous to those of Pascal and Brianchon in a Plane." (See *Journal*, vol. VI.

New Series, theorem (v.) p. 117, (xxii.) p. 131, and (xxiv.) p. 132). In that memoir this theorem is obtained by reciprocal polars, so that perhaps the following independent analytical proof may be interesting to some readers. Before proceeding with the investigation, however, I shall, to avoid unnecessary reference, enunciate the property, and I shall select M. Chasles's own enunciation because the form in which he presents the theorem is really that under which it is here proved.

The twelve tangent planes to a surface of the second degree, drawn through the edges of a tetrahedron, may be considered to intersect, three and three, in four points, each of which is the intersection of three planes drawn through edges in the same face of the tetrahedron; the four straight lines joining these points to the angles of the tetrahedron opposite to these faces will lie in an hyperboloid of one sheet and will belong to the same system of generators.

Let $t = 0, \quad u = 0, \quad v = 0, \quad w = 0,$

be the equations to the faces of the tetrahedron; then, supposing t, u, v, and w to have been multiplied by the proper constants, we may denote the equation to the surface by

$$t^2 + u^2 + v^2 + w^2 + 2\lambda tu + 2\mu tv + 2\nu tw + 2\rho uv + 2\sigma uw + 2\tau vw = 0 \quad \ldots\ldots\ldots (6).$$

Put
$$\alpha = 1 - \rho^2 - \sigma^2 - \tau^2 + 2\rho\sigma\tau,$$
$$\beta = 1 - \mu^2 - \nu^2 - \tau^2 + 2\mu\nu\tau,$$
$$\gamma = 1 - \lambda^2 - \nu^2 - \sigma^2 + 2\lambda\nu\sigma,$$
and
$$\delta = 1 - \lambda^2 - \mu^2 - \rho^2 + 2\lambda\mu\rho.$$

Equation (6) may be put under the form

$$(t + \lambda u + \mu v + \nu w)^2 + (1-\lambda^2)u^2 + (1-\mu^2)v^2 + (1-\nu^2)w^2 + 2(\rho - \lambda\mu)uv + 2(\sigma - \lambda\nu)uw + 2(\tau - \mu\nu)vw = 0.$$

Multiply this by $1 - \lambda^2$, and it readily takes the form

$$(1-\lambda^2)(t + \lambda u + \mu v + \nu w)^2 + \{(1-\lambda^2)u + (\rho - \lambda\mu)v + (\sigma - \lambda\nu)w\}^2 + \delta v^2 + \gamma w^2 + 2kvw = 0,$$

where k has a value which it does not concern us to know.

This form of equation (6) shews us that

$$\delta v^2 + \gamma w^2 + 2kvw = 0 \quad \ldots\ldots\ldots (7),$$

is the equation to the two tangent planes to the surface drawn through the edge (vw). But (7) can, by attributing a proper

value to r, be put under the form

$$-\frac{1}{r}(rv-\gamma w)(rw-\delta v)=0,$$

so that the equations to the two tangent planes are

$$rv=\gamma w \text{ and } rw=\delta v;$$

and in a similar manner the equations to the other tangent planes may be got. If we collect the whole and place in the same horizontal line the equations to those planes that intersect in one of the points mentioned in the theorem, we obtain

$$\left.\begin{array}{lll} lt=au, & mt=av, & nt=aw \\ lu=\beta t, & pu=\beta v, & qu=\beta w \\ mv=\gamma t, & pv=\gamma u, & rv=\gamma w \\ nw=\delta t, & qw=\delta u, & rw=\delta v \end{array}\right\} \ldots\ldots (8).$$

Eliminate t, u, v, and w from the first, second, third, and fourth rows of equations respectively, and we obtain the equations to the four straight lines referred to in the enunciation of the theorem.

$$\left.\begin{array}{l} \dfrac{u}{l}=\dfrac{v}{m}=\dfrac{w}{n} \\ \dfrac{t}{l}=\dfrac{v}{p}=\dfrac{w}{q} \\ \dfrac{t}{m}=\dfrac{u}{p}=\dfrac{w}{r} \\ \dfrac{t}{n}=\dfrac{u}{q}=\dfrac{v}{r} \end{array}\right\} \ldots\ldots\ldots\ldots (9).$$

Now it is easy to see that each of these lines lies in the hyperboloid whose equation is

$$(mq-np)(rtu+lvw)+(np-lr)(qtv+muw)$$
$$+(lr-mq)(ptw+nuv)=0 \ldots\ldots\ldots (10),$$

and thus the theorem is established.

Yorktown, near Bagshot,
May 19, 1851.

A GEOMETRICAL PROPERTY OF CURVES OF THE THIRD ORDER.

By Thomas Cotterill, M.A.,
Late Fellow of St. John's College, Cambridge.

The equation to a curve of the third order referred to a triangle by the usual coordinates $\alpha\beta\gamma$, and passing through two of its angular points, can be put under the form

$$\alpha\beta(A\alpha + B\beta) = \gamma f_2,$$

$f_2 = 0$ denoting a conic.

The form of this equation at once points out nine points on the curve of the third order (which for shortness we will call the curve); viz. the six intersections of the lines α, β, and $A\alpha + B\beta$ with the conic f_2, and the three intersections of the same lines with γ.

Now consider the four points in which $\alpha\beta$ cut the conic, and the point in which $A\alpha + B\beta$ meets γ, and observe that the equation to the curve can be put under the form

$$\alpha\beta(A\alpha + B\beta + \lambda\gamma) = \gamma(f_2 + \lambda\alpha\beta).$$

Here we have a conic passing through the first-mentioned four points and a line through the last-mentioned point; and it appears from the form of the equation just given, that the conic and line cut each other in points on the curve. Hence

"Every conic passing through four given points on a curve of the third order will cut it again in two points, such that the straight line joining them will pass through a fixed point on the curve."*

This point, from its characteristic property, we may call the focus of the inscribed quadrangle with respect to the curve.

If we except the case of a double point, this theorem is general. The points may be supposed to coalesce and the curve of the third order may break up into a conic and line, or three lines; and the conic passing through the four points may be any of the three systems of lines passing through the four points.

There are but few descriptive properties of curves of the third order given by Maclaurin, and subsequent writers, which cannot be proved by this theorem, and I think it will be found a powerful and easy instrument in discovering new

* The theorem may also be shewn as follows:

"If the four fixed points be the points of intersection of the conics $U = 0$ and $V = 0$, then the equation of the curve is $pU + qV = 0$, and that of an indeterminate conic is $aU + bV = 0$; the line forming the remaining two points of intersection of the curve and conic is $bq - ap = 0$, which passes through the point $p = 0$, $q = 0$.

relations. Its simplest corollaries point out the properties of the points of intersection of the common chords of the conic and curve with the curve,—the means of drawing a tangent to the curve at any point,—the conditions of contact of conics (drawn according to given laws) with the curve.

Let the curve of the third order consist of a conic and line. We may take three points on the conic and one on the line, and then apply the theorem.

Again, take two points on the conic and two on the straight line. Draw two lines joining the points on the conic with the points on the line, the remaining chord of intersection will cut the line in the focus; and similarly for the other two lines joining the points. This gives us a converse of Pascal's Theorem, but I am not aware that the more general property of the chord of intersection of the fixed conic and the variable conic passing through the fixed points has been observed.

Now, taking the general case, the conic passing through four points and their focus meets the curve again at the point where the tangent at the focus meets it. Hence if curves of the third order be drawn through five points so that one of them is always the focus of the rest with respect to all the curves, the tangent at this point to each curve will meet the curve on the conic passing through the five points.

Take two sets of four points and their foci, and consider the point where the line joining the foci again meets the curve. The conic through the first set of points and the second focus will pass through this point, as well as the conic through the last set of points and the first focus. But Plücker has shewn, in his "Theorie der Curven," (and it follows easily from Mr. Weddle's paper in the last number of this *Journal*), that it is also the point in which curves of the third order passing through the eight points intersect.

Hence we have the following remarkable theorem:

Through eight points draw a curve of the third order; we shall in general have 70 foci, and any one focus of four points has a corresponding focus of the remaining four. Then the 35 lines joining the corresponding foci pass through the same point on the curve, viz. the point in which all curves of the third order through the eight points intersect.

I think this property in the hands of abler geometers than myself will give a construction for determining the ninth point by purely descriptive linear methods. The analytical expressions for determining its position are not intricate, but involve some functions which require geometrical interpretation. I may however mention that one result is, that if the

points 123, 456 are such that each of the three sets of points 162, 243, 351 is in a straight line, then the curves of the third order passing through these points and two other points 78 will again intersect in the common point of intersection of the conics (16578), (24678), and (35478).

The equations on which these remarks are founded were supposed to contain point-coordinates and conics passing through points; but as these can be changed into line-coordinates and conics touching lines, we have the following reciprocal theorem:

"Every conic touching four tangents to a curve of the third class will be touched again by two of its tangents, and their point of intersection will describe another tangent to the curve."

One of the most interesting curves of this class is that touched by the line joining the feet of the perpendiculars from any point in a circle on the sides of an inscribed triangle.

The tangents at its cusps meet in the intersection of the perpendiculars from the angular points of the triangle on the opposite sides. If α, β, γ be the perpendiculars from the angular points of the inscribed triangle, whose angles are A, B, C, on a tangent to the curve, the tangential equation to the curve will be

$$\alpha(\beta-\gamma)^2 \cot A + \beta(\gamma-\alpha)^2 \cot B + \gamma(\alpha-\beta)^2 \cot C = 0.$$

London, March 12, 1851.

GEOMETRICAL PROPOSITIONS RELATING TO FOCAL PROPERTIES OF SURFACES AND CURVES OF THE SECOND ORDER.

By JOHN WALKER.

MANY geometers have at different times endeavoured to obtain from the cone itself a construction for the foci and directrices of Apollonius's sections of an oblique cone which should serve for studying their properties from elementary principles. The following general property of an oblique cone leads to a simple method of effecting this object.

If a plane cut a cone in a conic, the planes of two subcontrary sections of the cone in two right lines, and the sphere passing through these circular sections in a circle, the rectangle under the segments of a chord (or secant) of the circle drawn from any point in the conic is to the rectangle under the perpendiculars let fall from the same point on the

two right lines in the constant ratio of the products of the sines of the angles which the plane of the conic and any side of the cone respectively make with the planes of circular section.

To prove this, from any point in the conic and from the vertex of the cone let fall perpendiculars on the planes of circular section; the four perpendiculars will lie in two planes passing through the side of the cone drawn through the point in the conic, and cutting each one of the two planes of circular section in right lines, the segments of which, together with the segments of the side of the cone and the four perpendiculars, form the sides of four right-angle triangles which are similar two and two: from compounding the proportions among the sides of these it easily appears, that the rectangle under the intercepts on the side of the cone between the point and the planes of circular section is to the rectangle under the perpendiculars let fall from the point on those planes, as the rectangle under the segments of the side between the vertex and the same planes is to the rectangle under the perpendiculars let fall from the vertex of the cone. But the latter of these two ratios is equal to the product of the reciprocals of the sines of the angles which any side of the cone makes with the planes of circular section, while the second term of the proportion is equal to the product of the perpendiculars let fall on the lines in which the plane of the conic intersects the same planes multiplied by the product of the sines of the angles it makes with them; also the first term, being the rectangle under the segments of a chord of the sphere, is equal to the rectangle under the segments of a chord (or secant) of the circle, in which the plane of the conic intersects the sphere, drawn from the same point: whence the proportion enunciated in the above theorem is easily inferred. For brevity's sake the circle and pair of right lines may be called a conjugate focal circle and pair of right lines, as it will shortly appear that they are merely a particular case of a pair of conjugate focal conics similarly related to the given conic; the constant ratio (which evidently remains unchanged for any given section of the cone, however the sphere and the circular sections be changed) may be called the modulus of the conic with respect to the cone. When the radius of a focal circle is evanescent it may be called a focal point; and when a pair of focal right lines become coincident, the single right line in which they merge may be called a directrix. The following corollaries from the theorem above will be evident:

(1). If the plane of the conic touch the sphere, the point of contact will become a focal point, and the square of the line drawn from any point in the conic to it is to the rectangle under perpendiculars let fall on the conjugate focal pair of right lines in the constant ratio of the modulus of the conic to unity.

(2). When the plane of the conic, passing through the line of intersection of the circular sections of the cone, cuts the sphere passing through them in a circle, the tangent* drawn from any point on the conic to this focal circle is to the perpendicular let fall on the conjugate directrix in the subduplicate ratio of the modulus to unity.† (In such a case of course the conic is one of Apollonius's sections.)

(3). And if the plane of the conic touch the sphere, the point of contact and conjugate focal line become a focus and directrix of the conic.

(4). A directrix is the chord of the double contact which its conjugate focal circle has with the conic if it meet it;

(5). Or, more generally, a pair of focal right lines will be the chords in which the conjugate focal circle intersects the conic if it meets it.

(6). If there be two focal circles described, conjugate respectively each to one of a pair of directrices, the sum or difference of tangents drawn to them from any point in the conic will be constant; and the cosines of the angles which lines drawn from the centres of these circles to any point in the conic make with the tangent at the point, are as the cosines of the angles which the same lines make with the tangents to the circles drawn from that point; also the

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* To the length of a tangent drawn from an external point to the circle corresponds of course the half of the chord of the circle bisected at the point in case it fall within the circle.

† If the plane of the conic be perpendicular to the plane of the centres of circular sections, and be intersected by it in the line ACA', while any circular section is intersected by the same plane in the line BCB' (AA', BB' being the points in which these lines meet the cone, and C their point of intersection), and, D being taken on AA' so that rect. $A'CD = B'CB$, the line DB be drawn, it will easily appear that the angles ACB, ABD are respectively equal to the angles which the plane of the conic makes with the planes of circular section, and ABC, ADB to the angles which the side of the cone AB makes with the same planes, and hence that the modulus of the conic is equal to the ratio $AD : AC$, or $AD.A'C : ACA'$, or $ACA' \mp BCB' : ACA'$, or $a^2 \mp b^2 : a^2$, if a be half the axis AA', and b the semi-axis perpendicular to AA'; the upper sign being used if the conic be an ellipse, the lower if a hyperbola. If the conic be a parabola the modulus is in this case obviously equal to unity.

normal at any point divides the line joining the centres of the circles into segments whose ratio is equal to that of tangents to the circles drawn from the point; and when the radii of the circles become evanescent, in which case each coincides with a focus of the curve, these obviously reduce to the well-known properties of the lines joining any point in the conic with its foci: the radical axis of two such circles will obviously be parallel to, and equally distant from, their respective directrices, if these lines lie on opposite sides of the conic; and the points in the conic at which the sum of the lines drawn to the centres of the circles is a minimum are determined by describing the circle whose diameter is the line joining their centres of similitude.

(7). Generally, if the chord of a conic section passing through the centre of a focal circle be bisected at that point, the portion of it intercepted between the conjugate focal pair of right lines will be bisected at the same point; and if the point be on the conic the tangent at that point drawn between the focal right lines will be bisected at the point of contact.

(8). The polar of the point of intersection of a pair of focal lines with respect to their conjugate focal circle is also the polar of the same point with respect to the conic: hence if any point in the plane of the conic be given, the locus of the centres of focal circles (considering the conic as a section of different cones) conjugate to pairs of focal right lines drawn through that point is the perpendicular let fall from it on its polar with respect to the conic.

(9). And the focal point conjugate to a focal pair of right lines drawn through any given point in the plane of the conic is the foot of the perpendicular let fall from the given point on its polar with respect to the conic.

From the third of these corollaries is deduced the following construction for the foci of one of Apollonius's sections of an oblique cone: Let the plane drawn through V, the vertex of the cone, perpendicular to the planes of circular section, cut the cone in the sides VAB, $VA'B'$, the conic in the line AA', and the plane of any circular section in BB', and let AA', BB' intersect in C; from either end of AA', as A, take on that line lengths AE, AF, such that E, F may be harmonic conjugates with respect to A, A', and AE may be to AF in the subduplicate ratio of $ACA' \mp BCB'$ to ACA'; a problem of elementary geometry: through F draw a plane cutting the cone in a circle; it may easily be shewn that the square of EF is equal to the rectangle under the seg-

ments of a secant of the circle drawn from F, and hence that a sphere passing through this circle and the point E will touch the plane of the conic in that point and intersect the cone in a second circle, the plane of which will pass through the point F; the planes of the two circles, in which the sphere, which touches the plane of the conic in the point E, cuts the cone, will therefore intersect in a line lying in the plane of the conic, passing through the point F and perpendicular to the axis AA'; thus this line and the point E will be a directrix and focus of the conic; and if from A' lengths $A'E' = AE$, and $A'F' = AF$ be taken on AA', a line through F' perpendicular to AA' and the point E' will be the other directrix and its focus respectively. The upper sign in the above construction is to be used for an ellipse, (and for its applicability it is evidently necessary that AA' should be the major axis,) the lower for a hyperbola. If the conic be a parabola, the above construction evidently becomes indeterminate; but the points E, F may in this case be thus found: let A be the vertex of the parabola, and AB the diameter in which a circular section drawn through A is cut by the plane VAA'; join V with C, the middle point of AB, and draw from C a line making with CB an angle equal to the angle CVB, and meeting the side of the cone VB in D; the line DC produced, and a line from D parallel to AB, will meet the axis AA' in the points E, F: it is easy to shew that, if a circular section be drawn through F, a sphere described passing through this circle and through the point E will touch the plane of the parabola in that point, and will cut the cone in a second circle, the plane of which will pass through F; and therefore that a line through F in the plane of the parabola, perpendicular to AA', and the point E, will be the directrix and focus respectively of the parabola: it is also evident that this line and point are equally distant from the vertex A.

From the same corollaries it may be shewn that the focus is the only point in a conic possessing the property that all chords drawn through it have their poles lying on right lines drawn perpendicular to them from the point. Taking any point in the plane of the conic it may be considered as a focal point; if a chord drawn through it have its pole on a perpendicular line at the point, this pole must be the point of intersection of the focal pair of lines conjugate to the point; in order that this should be the case for all chords therefore, the point of intersection of the focal lines must be indeterminate, *i.e.* the lines must coincide; but the focus, as has been

shewn, is the only point for which the conjugate focal pair of right lines merge in a single directrix.*

But the theorem at the commencement of this paper is only a particular case of the following general theorem, which appears to contain the ultimate generalization of the whole theory of foci and directrices: to abbreviate the enunciation of it, it will be convenient to employ the term *conjugate focal conics* with respect to any given conic for two conics so related to it, that the rectangles under the segments of chords (or secants) of them respectively, drawn from any point in the given conic parallel to two fixed lines, have a constant ratio one to the other; and two surfaces of the second order similarly related to a given surface of the same order may be called conjugate focal surfaces with respect to it: these definitions being premised, the theorem may be thus stated: If three surfaces of the second degree have the same pair of curves of intersection (*i.e.* if any one of the three pass through the pair of curves in which the other two intersect), any plane meeting the three surfaces will cut them in three conics so related that any two are conjugate focal conics with respect to the third; and this is a consequence

* It may be objected to this proof, so elementary and general in its nature, that it necessitates the consideration of any given conic as cut from a variety of cones; for it is not of course possible to describe a sphere cutting a given cone in two plane circular sections and touching the plane of any given section of the cone in *any* point. It is easily seen geometrically that if spheres be described cutting a cone in circular sections, and cutting the plane of a given section in a circle of constant radius, the locus of the centre of this circle in the plane of the conic is, $P^2 \tan^2\theta + r^2 = mx^2 + ny^2$; x and y being coordinates with respect to certain axes whose origin is the foot of the perpendicular let fall from the vertex of the cone on the plane of the conic, P the perpendicular let fall on the line in which that plane intersects the plane of centres of circular sections, θ the angle between these planes, and r the constant radius of the circle, m and n being certain constants, depending on the angle θ, and other constant angles in the cone. The locus of points of contact, if the sphere touch the plane of the section, is of course found by putting $r = 0$, and the curve thus determined divides the plane of the conic into two regions; no sphere cutting the cone in circular sections can meet the plane of the conic in any point in one region, but every point in the other region is the centre of a circle in which the plane of the conic is intersected by some such sphere. It may possibly be not difficult to discuss this bounding curve (which of course is different for the same conic according as it is cut out of various cones) geometrically, as it is confocal with the conic; it is a hyperbola, parabola, or ellipse, according as the conic is an ellipse, parabola, or hyperbola; when the focal lines are at right angles it has also its semiaxes proportional to those of the given conic; and when the focal lines are parallel, it becomes that axis of the given conic which is perpendicular to them. The locus of the point of intersection of the focal pair of lines conjugate to the variable focal point is the curve which is reciprocal-polar to the locus of that point with respect to the given conic; the two points are *corresponding points* on their respective loci, and the line joining them is normal to the first locus.

of the analogous property of the surfaces themselves, viz. that any two will be conjugate focal surfaces with respect to the third. The geometrical proof of this theorem is as easy and elementary as the analytical, depending only on the well-known corollary from Desargues' theorem (itself, as has been often shewn, capable of the most elementary geometrical proof from the fundamental property of plane sections of a cone, viz. that the rectangles under the segments of chords (or secants) of a given conic drawn through any point in its plane, parallel to two fixed lines in that plane, bear a constant ratio one to the other), that if a transversal meet each of three conics, passing through the same four points, the six points of intersection will form an *involution*. The only definition of surfaces of the second order employed in the following proof is, that every section of them made by a plane is a conic; and the only property of conics employed is the fundamental one above alluded to, and the theorem of involution.

To fix the ideas, let the three surfaces be called S, S', S'', and L, L' the curves of their intersection; let a be a fixed point in S, abb' a fixed line meeting S' in the points bb', and acc' a fixed line meeting S'' in cc'; p being any variable point in S, let the line ap meet S' in the points rr' and S'' in the points $p'a'$; lastly let a line drawn from p parallel to abb' meet S' in qq', and a line drawn from the same point parallel to acc' meet S'' in the points kk'. Then $pq.pq' : ab.ab' = pr.pr' : ar.ar'$; but the latter ratio (by the corollary from Desargues' theorem stated above, applied to the system of three conics in which the plane of the lines pqq', abb' meets the three surfaces, which system of course pass through each of the four points in which the same plane meets the two curves of intersection of the three surfaces) is equal to the ratio $pp'.pa' : ap'.aa'$; this last ratio is obviously equal to the ratio $pk.pk' : ac.ac'$, therefore, finally, $pq.pq' : pk.pk' = ab.ab' : ac.ac'$.

If now the fixed lines abb' and acc' be supposed to be drawn in any plane cutting the three surfaces in three conics C, C', C'', and if from any point p on C a line be drawn parallel to abb' meeting C' in qq', and from the same point p a line parallel to acc', meeting C'' in kk', the rectangles $pq.pq'$ and $pk.pk'$ will bear a constant ratio one to the other.

A few particular cases of this theorem are easily deduced from its general form: (1). If three surfaces of the second order pass through the same curves of intersection, and if a plane touching one of the three at an umbilic cut the

others in two conics, the point of contact and either conic will be a conjugate focal point and conic to the second conic. (2). If two surfaces of the second order intersect in two plane curves, and any plane cut one of the two in a circle (or touch it at an umbilic), the planes of intersection in two right lines and the second surface in a conic, the circle (or point of contact) and the pair of right lines will be a conjugate focal circle (or point) and pair of right lines to the conic. (3). Or, if through the line in which the planes of the curves of intersection meet, a plane be drawn cutting the surface in two conics, either of these and the right line will be a conjugate focal conic and directrix to the other conic. (4). And if the tangent plane to one of the surfaces, drawn through the same line, should meet it at an umbilic, and cut the other surface in a conic, the point of contact and the right line will be a focus and directrix of the conic; and since every point on a sphere is an umbilic, (5) If through the line of intersection of two planes, which cut a surface of the second order in circles, a plane be drawn touching the sphere which passes through these circles and cutting the surface in a conic, the point of contact and right line will be a focus and directrix of the conic. (6). Also, if two surfaces of the second order envelope one the other along a plane curve, and any plane cut the surfaces in two conics, and the plane of their curve of contact in a right line, either conic and this right line are a conjugate focal conic and directrix to the other conic. (7). And the plane which touches one surface at an umbilic cuts the other surface in a conic, and the plane of the curve of contact in a right line, such that the point of contact and the right line are a focus and directrix of the conic; a theorem given by M. Chasles, in the *Annales de Gergonne*, tom. XIX. p. 167.

It is obvious that if three conics be so related that two of them are conjugate focal conics to the third, any transversal will meet the three in six points forming an involution; also that if three conics be so related that any transversal meets them in six points which form an involution, then any two of the conics will be conjugate focal conics with respect to the third: hence it appears that if three conics be so related that two of them are conjugate focal conics to the third, then this third and either of the two will be conjugate focal conics to the remaining one of the three.

It is also evident from the nature of the proof of the theorem respecting the surfaces S, S', S'', that if (instead of the hypothesis that the third passes through the curves of

intersection of the other two) they had been supposed to be so related, that any transversal met them in six points forming an involution, it would equally have followed that any two were conjugate focal surfaces with respect to the third; and conversely, &c. &c.

The theorem respecting the surfaces S, S', S'' may easily be thrown into the following form, when S', S'' are central surfaces: Let t, t' be the lengths of tangents drawn from any point in S to S', S'' respectively; let r be a semi-diameter of a certain surface M, similar to and similarly placed as S', parallel to t, and r' a semi-diameter of a certain surface M', similar to and similarly placed as S'', parallel to t'; then $t : t' = r : r'$. In this form the theorem leads synthetically to the following method of generating a surface S of the second order, from two given central surfaces of the second order* S' and S''. Let M and M' be two surfaces respectively similar to and similarly placed as these, then the locus of a point such that, tangents t, t' being drawn from it to $S'S''$, and semi-diameters r, r' of M, M', respectively parallel to these tangents, t is to t' as r to r', is a third surface of the second order S; and the three surfaces $SS'S''$ will be so related that any two will be conjugate focal surfaces to the third. To shew that the *modular* and *umbilicar* methods of generating surfaces of the second order are particular cases of this one general method, it will be necessary in the present state of the geometry of such surfaces to call in the aid of conceptions arising from analytical rather than geometrical methods.

It may be shewn that if a sphere S' and a finite surface of the second order S'' intersect a given surface S of the second order in the same pair of curves (real or imaginary), and if S' and S'', remaining constant in species, diminish in dimension without limit, then the points s', s'', into which they degenerate, become ultimately any *corresponding points* on a certain surface confocal with S, and its reciprocal polar with respect to S, respectively; if S'' be supposed to be a hyperboloid, then, S remaining constant, the same will be true of s', considered as the limit of the sphere, and s'' the ultimate position of the vertex of the cone into which S'' degenerates. To shew these things, let

$$Ax^2 + By^2 + Cz^2 = 1$$

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* It is obvious that if $f(xyz)$, $f_1(xyz)$ be any two rational and integral functions of the n^{th} degree, the geometrical interpretation of the equations $f(xyz) = 0$, $f_1(xyz) = 0$, and $f(xyz) - \mu f_1(xyz) = 0$, suggests an analogous generation for surfaces of any order.

be the equation of the surface S, and

$$(x-x')^2+(y-y')^2+(z-z')^2=\rho'^2,$$
$$A''(x-x'')^2+B''(y-y'')^2+C''(z-z'')^2=\rho''^2,$$

be the equations to the sphere S' and the surface S'' respectively; the equations

$$Ax^2+By^2+Cz^2=1,$$

and $$(1-mA'')\,x^2+(1-mB'')\,y^2+(1-mC'')\,z^2$$
$$-2(x'-mA''x'')\,x-2(y'-mB''y'')\,y-2(z'-mC''z'')\,z$$
$$=\rho'^2-m\rho''^2-(x'^2+y'^2+z'^2)+m(A''x''^2+B''y''^2+C''z''^2),$$

must be identical, m being a certain constant; hence we have the equations

$$x'=mA''x'',\quad y'=mB''y'',\quad z'=mC''z'';$$
$$m=\frac{A-B}{AB''-BA''}=\frac{A-C}{AC''-CA''}.$$

Also

$$\frac{1-mA''}{A}=\rho'^2-m\rho''^2-(x'^2+y'^2+z'^2)+m(A''x''^2+B''y''^2+C''z''^2).$$

Supposing ρ' and ρ'' to diminish without limit, and eliminating x'', y'', z'' from this equation, we have for the locus of $x'y'z'$

$$(1).\quad 1=\frac{A}{mA''}x'^2+\frac{B}{mB''}y'^2+\frac{C}{mC''}z'^2,$$

a surface which, by the substitution of the values of m given above in terms of ABC, $A''B''C''$, may easily be shewn to be confocal with the surface

$$1=Ax^2+By^2+Cz^2;$$

similarly, eliminating x', y', z' from the same equation above, we have, as the locus of $x''y''z''$, the surface

$$(2).\quad 1=mAA''x''^2+mBB''y''^2+mCC''z''^2.$$

The surfaces (1), (2) are obviously polar-reciprocal with respect to S; and the equations connecting $x'x''$, $y'y''$, and $z'z''$, shew that the points $x'y'z'$ and $x''y''z''$ are *corresponding points* on those surfaces.

From the foregoing analysis it also appears (by putting $y'=0$, supposing the axis of y to coincide with the mean axis of S, and also $B''=0$, $\rho''=0$), that if a sphere S' intersect a given surface S in two plane sections L, L', and if S' be supposed to diminish without limit, the point s',

which is the ultimate position of S', and the point in which the line of intersection of the ultimate position of the planes LL' pierces the plane of the greatest and least axis of S, will be corresponding points on the *excentric conic* of S in that plane, and its reciprocal polar with respect to the section of S by the same plane. The readers of the *Journal* are well aware of the many applications of this latter principle which Mr. Townsend has made in its pages.

Returning, after these preliminary Lemmas, to the proportion $t : t' = r : r'$, it will at once appear from the manner in which this is deduced that, the surface S remaining constant, S' and S'' will diminish in the same ratio, and hence that the surfaces M, M' remain constant; so that, if S' be supposed to be a sphere, and K a certain constant, and l, l' lines drawn from any point in S to the points s', s'', which are the ultimate positions of the sphere S' and the surface S'' respectively, and if r' be a semi-diameter of M parallel to l', then $l : l' = K : r'$, which is the form of Mr. Willock's generalization of MacCullagh's *modular* generation of surfaces of the second order.

If, again, in the same general theorem respecting $SS'S''$, S' be supposed to be a sphere and S'' the system of two planes L, L' in which it intersects S, and if from any point p in S perpendiculars pq, pq' be let fall on the planes L, L' respectively, and a tangent pt be drawn to S', then pt^2 is to $pq.pq'$ in a constant ratio, and, proceeding to limits, this theorem leads synthetically to Mr. Salmon's *umbilicar* method of generation. It is not uninteresting, perhaps, to observe how the *modular* and *umbilicar* methods flow synthetically from one general theorem expressing the conjugate focal relation of any two with respect to the third of three surfaces $SS'S''$ connected by the relation of the involution of the six points in which they are met by any transversal, the geometrical relation which answers to the analytical one of the three surfaces having the same real or imaginary curves of intersection. Perhaps had the ideas suggested by the relation of involution been pursued farther, it might have been unnecessary to introduce the analytical method of proof to establish the connecting link between the general theorem of the three surfaces $SS'S''$ and the *modular* and *umbilicar* methods of generating surfaces of the second order.

To these methods may evidently be added the following one, from what has been established above: If s' be a point, and S'' a cone of the second order, and if a point p be taken such that, joining ps' and drawing from p a line pqq', parallel to a fixed line and meeting the cone in qq', ps'^2 always bear

a constant ratio to the rectangle $pq.pq'$, p will generate a snrface S of the second order, and the ratio will remain constant for the same surface S if s' move in a surface confocal with S, and the vertex of the cone (the cone moving parallel to itself) move so as always to be at the point *corresponding* to s' of the surface which is the reciprocal polar of the locus of s' with respect to the same surface S. The *umbilicar* method is obviously the particular case of this in which the cone breaks up into two planes.

From the same general theorem of the three co-intersecting surfaces $SS'S''$ many other theorems may be obviously deduced, such as that, *e.g.*: If two planes touch a surface of the second order in two points, the rectangle under perpendiculars let fall from any point on the surface on the planes is to the square of the perpendicular let fall on the chord of contact in a constant ratio; or, more generally, if two surfaces have double contact, and if through any point of one a chord (or secant) of the other be drawn parallel to a fixed line, and from the same point a perpendicular be let fall on the chord of contact, its square will bear a constant ratio to the rectangle under the segments of the chord (or secant).

The value of the ratio which has been called in the commencement of this paper the *modulus* of a section of a cone, may in the general case be expressed in terms of half the angle (i) between the right lines in which the plane of the section is intersected by the planes of circular section, and of the semi-diameters (a, b) parallel to the internal and external bisectors of that angle, by the formula

$$a^2 \mp b^2 : a^2 \sin^2 i \pm b^2 \cos^2 i,$$

if the conic be an ellipse or hyperbola; and by the formula $1 : \sin^2 i$, if it be a parabola. This value of the modulus may be thus obtained: Let the plane drawn through V, the vertex of the cone, perpendicular to the planes of circular section, intersect the plane of the conic in the line AA', and the planes of two circular sections, drawn through O, any point in AA', in the lines BB', CC'; and let the planes of the circles intersect the plane of the conic in the lines mOm', nOn', the points AA', BB', CC', mm', nn', being the points in which these lines meet the cone; and let any sphere described with the point O as centre be met by the lines Om, On, OA, OB, OC, and by the line of intersection of the planes of the circular sections drawn through O, in the six points $\theta, \theta', f, g, h, \theta''$, so that $\theta\theta'\theta''$ are the angles of the spherical triangle formed by the planes of the conic and

of the two circular sections, and the arcs $\theta''f$, $\theta''g$, $\theta''h$, are quadrants. If ϕ, ϕ' be the angles which the side of the cone $VABC$ makes with the planes of circular section (*i.e.* the angles VBB', VCC'), we shall obviously get, as in note p. 18,

$$\sin AOB.\sin AOC : \sin\phi \sin\phi'$$
$$= AOA' \mp COC' : AOA' = AOA' \mp mOm' : AOA' :$$

but from the spherical triangles,

$$\sin AOB = \sin fg = \sin\theta\theta''f = \sin\theta \sin f\theta = \sin\theta.\sin mOA ;$$

similarly,

$$\sin AOC = \sin fh = \sin\theta'\theta''f = \sin\theta' \sin f\theta' = \sin\theta' \sin nOA ;$$

therefore the proportion above becomes

$$\sin\theta.\sin\theta'.\sin mOA.\sin nOA : \sin\phi \sin\phi' = AOA' \mp mOm' : AOA' ;$$

but
$$AOA' \mp mOm' : AOA'$$
$$= \sin mOA.\sin nOA\,(a^2 \mp b^2) : a^2 \sin^2 i \pm b^2 \cos^2 i,$$

by the properties of the conic; therefore finally

$$\sin\theta \sin\theta' : \sin\phi \sin\phi' = a^2 \mp b^2 : a^2 \sin^2 i \pm b^2 \cos^2 i.$$

Kingstown, near Dublin,
April 9, 1851.

ON PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

By Professor De Morgan.

The reasons why I think it worth while to present the following investigation, will appear from the historical remarks which follow it.

As usual, let
$$dz = p\,dx + q\,dy \ldots\ldots\ldots\ldots\ldots (1),$$

and let
$$\phi(x, y, z, p, q) = 0 \ldots\ldots\ldots\ldots (2),$$

be a partial differential equation of which the complete solution is required. We must determine the form of ψ in

$$\psi(x, y, z, p, q) = 0 \ldots\ldots\ldots\ldots (3),$$

in such manner that p and q, obtained in terms of x, y, z, from (2) and (3), and substituted in (1), will render (1) integrable by means of a factor.

Let this factor be m. We have then

$$\frac{dm}{dx} = -\frac{d(mp)}{dz}, \quad \frac{dm}{dy} = -\frac{d(mq)}{dz}, \quad \frac{d(mp)}{dy} = \frac{d(mq)}{dx} \ldots (4),$$

$$q\frac{dp}{dz} - p\frac{dq}{dz} + \frac{dp}{dy} - \frac{dq}{dx} = 0 \ldots\ldots\ldots (5).$$

Though the single equation (5) is not coextensive with the system (4), nevertheless, (5) being satisfied, (1) can be integrated. To shew this, first let z be constant, and let λ be the factor which *then* makes $pdx + qdy$ integrable: let

$$\lambda p dx + \lambda q dy = dt.$$

Then, making z again variable, we have

$$dt = \left(\lambda + \frac{dt}{dz}\right) dz \ldots\ldots\ldots\ldots (6).$$

This equation will contain only t and z, if

$$\frac{dt}{dy} \cdot \frac{d}{dx}\left(\lambda + \frac{dt}{dz}\right) - \frac{dt}{dx} \cdot \frac{d}{dy}\left(\lambda + \frac{dt}{dz}\right) \ldots\ldots (7)$$

vanish. By aid of $\frac{dt}{dx} = \lambda p$, $\frac{dt}{dy} = \lambda q$, $\frac{d(\lambda p)}{dy} = \frac{d(\lambda q)}{dx}$, we shall find (7) reducible to λ^2 multiplied by the first side of (5). Consequently, when (5) is satisfied, (6) is a common differential equation, which gives t in terms of z. But t is already known in terms of x, y, z, so that a relation can be found between x, y, z, without t, which is an integral of (1). That is, if p and q be any functions of x, y, z, which render (5) identical, they also render (1) integrable: and the converse.

It follows, then, that to determine every integral of (2), we may determine every form of ψ in (3) which, with the given form of ϕ in (2), implies such values of p and q in terms of x, y, z, as have just been described. Now from (2) and (3) we easily obtain (denoting partial differentiation by variables suffixed)

$$\frac{dp}{dx} = \frac{\phi_q\psi_x - \phi_x\psi_q}{\phi_p\psi_q - \phi_q\psi_p} \quad \frac{dq}{dx} = \frac{\phi_x\psi_p - \phi_p\psi_x}{\phi_p\psi_q - \phi_q\psi_p} \ldots\ldots (8),$$

in which x may be changed into y or into z. Substitution in (5) gives

$$(\phi_x + \phi_z p)\,\psi_p + (\phi_y + \phi_z q)\,\psi_q = (\psi_x + \psi_z p)\,\phi_p + (\psi_y + \psi_z q)\,\phi_q \ldots (9),$$

from which, ϕ being given, ψ is to be determined, or, ψ being given, ϕ is to be determined. Making the first supposition, let $d\phi = Xdx + Ydy + Zdz + Pdp + Qdq$: we have then

$$P\frac{d\psi}{dx} + Q\frac{d\psi}{dy} + (Pp + Qq)\frac{d\psi}{dz} = (X + Zp)\frac{d\psi}{dp} + (Y + Zq)\frac{d\psi}{dq} \ldots (9)'.$$

From (9)' ψ must be determined by help of the simultaneous system

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{Pp + Qq} = -\frac{dp}{X + Zp} = -\frac{dq}{Y + Zq} = \frac{d\psi}{0} \quad \ldots(10).$$

Of this system ϕ = const. and ψ = const. are both primitives. If we can determine three others, independent of the two just named, A = const., B = const., C = const., then the complete integral of (9)' is $\psi = f(A, B, C, \phi)$, where f is any form whatever. That is, the complete solution of $\phi = 0$ is to be found by integrating $dz = pdx + qdy$, where p and q are determined from

$$\phi = 0, \quad f(A, B, C) = 0 \ldots\ldots\ldots(11).$$

The above is substantially the procedure of Lagrange and Charpit, as presently noticed, made perhaps clearer by the direct use of the criterion in (7), and shorter by that of the equations in (8). The great difficulty of the result, on which I cannot get a satisfactory view from anything which has been written, is that which arises from its apparently too great generality. We know that $\phi = 0$ can only introduce into its solution an arbitrary relation between *two* given forms, while (11) exhibits an arbitrary relation between *three*. Undoubtedly, in any case which might be proposed, the actual attainment of the final result from $dz = pdx + qdy$ would shew that this excess of generality is only apparent: but this does not satisfy the inquirer.

The difficulty appears to me to have its source in a neglect of the conditions of the problem. When we demand the solution of the equation $\phi(x, y, z, p, q) = 0$, we require not only that $\phi = 0$ shall exist, but that $dz = pdx + qdy$ shall exist, and not only exist, but exist in a particular mode, so as to admit of integration. The system (11) satisfies only the first and third conditions: for (1) and (9)' neither express nor imply $dz = pdx + qdy$, but only provide the equation to be integrated, and the means of satisfying the condition without which $dz = pdx + qdy$ cannot exist *in the manner proposed*. Consequently, the express introduction of this last equation is really a new condition, and a limitation of generality. But as we cannot verify this last assertion directly, at least while f is perfectly general, we must endeavour to succeed by other than direct means. That is, instead of applying (11) to (1), we must apply (1) to (11).

Assuming $dz = pdx + qdy$, we have

$$dA = (A_x + A_z p)\, dx + (A_y + A_z q)\, dy + A_p dp + A_q dq,$$

and similar forms for dB, dC, $d\phi$. Multiply by θ, λ, μ, ν, and add, assuming the ratios of θ, λ, μ, ν, so as to satisfy

$$\left.\begin{aligned}(A_x + A_z p)\,\theta + (B_x + B_z p)\,\lambda + (C_x + C_z p)\,\mu + (\phi_x + \phi_z p)\,\nu = 0\\ (A_y + A_z q)\,\theta + (B_y + B_z q)\,\lambda + (C_y + C_z q)\,\mu + (\phi_y + \phi_z q)\,\nu = 0\\ A_q\quad\theta + \quad B_q\quad\lambda + \quad C_q\quad\mu + \quad \phi_q\quad\nu = 0\end{aligned}\right\}$$

.....(12).

And thus we shall have

$$\theta dA + \lambda dB + \mu dC + \nu d\phi = (A_p\theta + B_p\lambda + C_p\mu + \phi_p\nu)\,dp \ldots(13).$$

But since A = const. is a solution of (9), we have

$$(A_x + A_z p)\,\phi_p + (A_y + A_z q)\,\phi_q = (\phi_x + \phi_z p)\,A_p + (\phi_y + \phi_z q)\,A_q,$$

and similar equations from B = const., C = const., ϕ = const. Multiply by θ, λ, μ, ν, and add, and we see, by use of (12), that

$$A_p\theta + B_p\lambda + C_p\mu + \phi_p\nu = 0 \ldots\ldots\ldots (12)',$$

so that

$$\theta dA + \lambda dB + \mu dC + \nu d\phi = 0 \ldots\ldots\ldots (14).$$

This equation merely implies $dz = p\,dx + q\,dy$, to which it is reduced by substituting for A, B, C, ϕ, their values in terms of x, y, z, p, q. It cannot therefore in any case be integrable *per se*. If we now introduce the condition $\phi = 0$, we have

$$\theta dA + \lambda dB + \mu dC = 0 \ldots\ldots\ldots (15).$$

In this, by aid of A, B, C, known in terms of x, y, z, p, q, and of $\phi = 0$, we can determine x, y, z, q, in terms of A, B, C, p, and substitute them in θ, λ, μ. But (15) cannot be integrable *per se*, even after this substitution: for as it then amounts to $\phi = 0$, $dz = p\,dx + q\,dy$, we should in that case have this last equation integrable whatever function of x, y, z might be taken for p, provided only that q be a certain determinate function of x, y, z, and p. Nevertheless, in the process which is to follow, it matters nothing whether (15) be integrable or not *per se*, on account of the entrance of the arbitrary condition $f(A, B, C) = 0$, from (11). The difference between the two cases, as explained by Monge, is as follows. When (15) is integrable, it determines a relation between A, B, C, which exists whatever other relation may happen also to exist. But when (15) is not integrable, it only determines the manner in which the relation which is to exist shall depend upon what other relation may happen to exist.

Now we know, (11), that we have to assume $C = f(A, B)$, at pleasure. If we substitute for dC in (15), we obtain the form $dA + M\,dB = 0$, where M may be made a function of A, B, p, at most. If p exist in this last equation, it must

be a function of A and B. Assume $p = \varpi(A, B)$, and then we have an ordinary differential equation, from which arises, say $F(A, B) = 0$. From the three equations $C=f$, $p=\varpi$, $F=0$, we may express A, B, C, in terms of p. Now all the variables are expressed already in terms of A, B, C, p. Hence all are functions of p alone, or x is a function of y, which is absurd. The only alternative is, that (15) should not contain p, which is therefore established. And all this holds whether (15) be integrable or not.

Now the two equations (whether the second be integrable or not), $C = f(A, B)$, $\theta dA + \lambda dB + \mu dC = 0$,

in which f is arbitrary, and λ and μ functions of A, B, C only, are easily shewn to be equivalent to the following. Let $B = FA$, $C = fA$, subject to the condition

$$\theta + \lambda F'A + \mu f'A = 0,$$

in which either f or F is taken at pleasure, and the other determined. But this also contains the complete integration of $\phi = 0$: the solution of which is obtained by eliminating p and q between the three equations

$$\phi = 0, \quad B = FA, \quad C = fA,$$

A, B, C, being determinate functions of x, y, z, p, q.

I take Lagrange's example, $pq - z = 0$, or $\phi = pq - z$. We have then

$$\frac{dx}{q} = \frac{dy}{p} = \frac{dz}{2pq} = -\frac{dp}{-p} = -\frac{dq}{-q} \quad \ldots\ldots (10);$$

whence, besides $pq - z = 0$, we get $q = Ap$, $x - q = B$, $y - p = C$. Now the equation (15), though deduced from the junction of $dz = pdx + qdy$ with the other forms, in our general survey, may here be most conveniently obtained by direct application of these forms. Doing this, we have

$$d(pq) = pd(q + B) + qd(p + C),$$

or $$pdB + qdC = 0, \quad \text{or} \quad dB + AdC = 0 \ldots . (15).$$

Assume $B = FC$, $A = fC$, and $F'C + fC = 0$; and obtain z in terms of x and y by eliminating p and q between

$$z = pq, \quad \frac{q}{p} = -F'(y - p), \quad x - q = F(y - p).$$

It will be observed that we begin by seeking and finding a method of rendering (1) integrable, supposing that we shall then have to find the integrating factor, *if we can:* but that

we end by a process which produces a solution out of the satisfaction of the criterion, without any apparent need of attempting the main investigation, to which the satisfaction of the criterion was preliminary. At this end we arrive whenever we can completely solve the system (10). But in many cases it will happen that we can only find a partial solution, such as $A = a$. All we can then do is to determine p and q from $\phi = 0$, $A = a$, and to integrate $dz = pdx + qdy$: by which we produce a *primary solution*, having the two constants, a, and the one introduced in integration. This must be treated in the usual way, and its result is often more convenient than that derived from a more complete solution of (10). Thus, in the above example, if we take $q = a^2p$, $z = pq$, we find

$$dz = \sqrt{z}\left(\frac{dx}{a} + ady\right), \quad \sqrt{z} = \tfrac{1}{2}\left(\frac{x}{a} + ay\right) + b,$$

in which we make $b = \psi a$, &c.

Lagrange, in 1772, published in the *Berlin Memoirs* the root of the above method, depending directly upon (5), and using for the integration of (5) such methods as Euler and himself then had at command. A year or two afterwards* he gave, by itself, the now common method by which (9)′ is made to depend upon (10).

But Lagrange did not carry his method so far as to augment the number of independent variables, as done in the transformation of (5) into (9)′, by which the complete dependence upon linear form is established. This "rapprochement," as Lacroix calls it, or connecting step between two methods of Lagrange, was made by Charpit, a young man of high promise, whose death (Lacroix seems to insinuate) prevented his memoir from being printed (*Calc. Diff.*, vol. II. p. 548). The complete method thus bears the name of Charpit in Lacroix's work, which may have caused it to attract less attention than it would have done under that of Lagrange: nothing more than the original method, as it stood previously to the *rapprochement*, appears in Peacock's or Gregory's examples. When it was resumed by Lagrange, it was only with reference to an incident of the solution, as follows. Up to this time, nothing had been contemplated beyond using $\phi = 0$, and $f(A, B, C) = 0$, to determine p and q for substitution in (1), to be followed by actual integration.

* I cannot refer to this paper: Monge and Lacroix give it the date 1779, D. F. Gregory gives 1774 as well.

But this gave too much generality for the solution of an equation of the first order with *two* independent variables; there being *three* definite forms in the arbitrary function. Lagrange (Lacroix, vol. II. p. 564) set himself to remove this difficulty, which he succeeded in doing, and thereby, which he hardly appears to have noticed, he completed his solution, as well as removed an anomaly. But his reasoning is not altogether correct. Having shewn on general grounds that such an equation as (13), in which $d\phi = 0$, must exist, he then reasons (according to Lacroix) as follows, symbols and equation-references being altered to suit this paper:—"... puisque les équations $A = a$, $B = b$, $C = c$, desquelles il résulte $dA = 0$, $dB = 0$, $dC = 0$, vérifient l'équation (1), il faut que la partie multipliée par dp dans l'équation (13) s'anéantisse d'elle-même, et que l'équation (1) transformée par les variables A, B, C, se réduise à la forme (15), θ, λ, μ, étant des coefficiens dépendans de F_1, F_2, F_3." These last are the expressions for x, y, z, in terms of A, B, C, p: and p is then declared to disappear from (15) because the criterion already requires that there should be a relation (11) between A, B, C. This argument, if good for anything, would go as far as to prove that (15) must be the very equation which results from that relation; which it is not, and cannot be, (15) being unintegrable. The mode of satisfying an equation which cannot be integrated by help of a factor, and the geometrical meaning of the solution, was given by Monge in 1784, and up to that time such an equation, for instance, as $dz = dx + xdy$ was held to be a contradiction, as Monge distinctly states.

The last-mentioned analyst, who, of all that ever lived, had the greatest power of eliciting the geometrical meaning of a formula, deduced the whole method from the properties of surfaces. Not having seen the *rapprochement* of Charpit, which was first printed by Lacroix, he may easily have thought that geometry had given him a method beyond analysis, or at least beyond easy attainment. The *ten* equations which arise from all modes of equating the first five quantities in (10) are deduced in their *separate* meanings, as belonging to his celebrated *characteristics* of a surface, or to the other curves or to the developable surfaces which thence arise. This method first appeared either in the second or third (1807) edition of the *Application de l'Analyse, &c.*

M. Cauchy, in a memoir which (having only a separate copy) I can but conjecture to belong to the *Bull. de la Soc. Philomath.*, gives the above method complete, and extends

it (as indeed did Charpit) to the case of three independent variables; but does not seem to be aware of what Charpit and Monge had done. Poisson (*Bull. de la Soc. Philomath.* 1815, p. 183) has treated of the difficulty in overcoming which Lagrange completed his solution: but he appears to me to shew nothing more than that the apparently too general form fully contains the ordinary; one of which there could be no doubt: the difficulty lies in shewing that it does not contain more.

October 10, 1851.

ON THE LOGARITHMIC PARABOLA.

By the Rev. J. Booth, LL.D., F.R.S., etc.

It has long been known that plane curves of the second order may be rectified by circular arcs, by logarithms, or by elliptic integrals of the first and second orders. It is only very recently however that these geometrical types have been extended so as to embrace elliptic integrals of every order, and that it has been shewn that all elliptic integrals—the parameter ranging from infinity positive to infinity negative—represent the symmetrical intersections of surfaces of the second order. These curves, to distinguish them by an appropriate name, may be called Hyperconic Sections.

Some years ago, however, an extension of this theory was given, both in this country and on the continent. In the Mathematical Journals of that time,* demonstrations were given of the theorem, now well known, that an elliptic integral, of the third order and circular form, represents a spherical conic section, or a particular class of hyperconics.

In a work published in the early part of last year† the author has shewn that an elliptic integral of the third order and logarithmic form, represents the symmetrical intersection of a paraboloid of revolution with an elliptic cylinder, when the parameter is negative and of the form $i^2 \sin^2\theta$, i being the modulus; and that when the parameter is negative and greater than 1, or of the form $\operatorname{cosec}^2\theta$, the integral represents the intersection of a paraboloid of revolution with an hyperbolic cylinder. Thus the ordinary formulæ for the rectification

* *Liouville's Journal*, 1841. *Philosophical Magazine*, 1842.

† The Theory of Elliptic Integrals, and the Properties of Surfaces of the Second Order applied to the investigation of the Motion of a Body round a Fixed Point. London: G. Bell. 1851.

of the conic sections are merely particular cases of those more general expressions known as elliptic integrals. More than this, all, or nearly all, the forms of comparison of those functions may simply and easily be derived from the properties of those curves or hyperconic sections, as they may with propriety be termed, described as they are, not on a plane, but on a sphere or a paraboloid.

1. We propose here to treat of the rectification of one of those hyperconics, the logarithmic parabola; or the curve of intersection of a parabolic cylinder and a paraboloid of revolution; the vertex of this surface being supposed to touch at its focus the plane of the parabola, the base of the parabolic cylinder.

Let the equation of the paraboloid be

$$x^2 + y^2 = 2kz \quad \ldots\ldots\ldots\ldots\ldots (1),$$

and $y^2 = 4g^2 + 4gx$, that of the parabolic base of the cylinder, the origin being at the focus. k is the semiparameter of the paraboloid, and g is one-fourth of the parameter of the base.

$$\therefore\ y^2 + x^2 = = (2g + x)^2 = 2kz \quad \ldots\ldots\ldots\ldots (2),$$

hence, x being the independent variable,

$$\frac{dz^2}{dx^2} = \frac{(2g + x)^2}{k^2},\quad \frac{dy^2}{dx^2} = \frac{g}{g + x} \quad \ldots\ldots\ldots (3),$$

therefore

$$\frac{d\Sigma}{dx} = \frac{(2g + x)\,[k^2 + (g + x)(2g + x)]}{\sqrt{\{k^2(g + x)(2g + x)\,[k^2 + (g + x)(2g + x)]\}}} \quad \ldots\ldots (4).$$

Now the expression under the radical being a quadrinomial in x, must be reducible to the usual form of an elliptic integral. We must choose a suitable transformation. Let

$$\tan^2\mu = \frac{dz^2}{dx^2 + dy^2} = \frac{(2g + x)(g + x)}{k^2} \quad \ldots\ldots\ldots (5),$$

deriving this value from (3). Substituting this value in (4) and reducing, we obtain the simple expression

$$\frac{d\Sigma}{dx} = \frac{2g + x}{k \sin\mu} \quad \ldots\ldots\ldots\ldots\ldots (6).$$

μ is evidently the inclination to the plane of xy, of a tangent drawn to the curve.

We must now eliminate x. Since

$$k^2 \tan^2\mu = 2g^2 + 3gx + x^2,$$

adding and subtracting $2g^2 - gx$, we shall have

$$k^2 \tan^2\mu = (2g + x)^2 - g(2x + x).$$

Completing the square, and taking the square root,

$$2(2g + x) = g + \sqrt{(4k^2 \tan^2\mu + g^2)} \ldots\ldots\ldots (7).$$

The positive sign only must be taken, for when $x = -g$, $\tan\mu = 0$. Substituting this value of $2g + x$ in the expression for the arc,

$$\frac{d\Sigma}{dx} = \frac{g + \sqrt{(4k^2 \tan^2\mu + g^2)}}{2k \sin\mu} \ldots\ldots\ldots (8).$$

If now we differentiate (5), we shall obtain

$$\frac{dx}{d\mu} = \frac{2k^2 \sin\mu}{\cos^3\mu \sqrt{(4k^2 \tan^2\mu + g^2)}} \ldots\ldots\ldots (9).$$

Multiplying the last equation by this expression,

$$\frac{d\Sigma}{d\mu} = \frac{d\Sigma}{dx}\frac{dx}{d\mu} = \frac{gk}{\cos^3\mu \sqrt{(4k^2 \tan^2\mu + g^2)}} + \frac{k}{\cos^3\mu},$$

or $$\Sigma = k\int \frac{d\mu}{\cos^3\mu} + gk \times \int \frac{d\mu}{\cos^3\mu \sqrt{(g^2 + 4k^2 \tan^2\mu)}} \ldots (10)$$

There are now three cases to be considered:

$$2k = g, \quad 2k < g, \quad 2k > g.$$

Case I. Let $g = 2k$, and the last equation will become

$$\Sigma = k\int \frac{d\mu}{\cos^3\mu} + k\int \frac{d\mu}{\cos^2\mu} = k\int \frac{d\mu}{\cos^3\mu} + k\tan\mu \ldots (11).$$

Now $k \tan\mu$ is the ordinate of a parabola, and $k\int \frac{d\mu}{\cos^3\mu}$ is the length of an arc of this parabola from the vertex to a point where a tangent to it makes the angle μ with the ordinate. Hence if we assume on the logarithmic parabola a point M, and through this point draw a plane touching the parabolic cylinder, this plane will be vertical, and will cut the vertical paraboloid in a parabola, whose semiparameter will be k. This parabola will touch the logarithmic parabola at the point M. Hence in this case the length of the logarithmic parabola to the point M, will be equal to the arc of the plane parabola from its vertex to the point M, plus the ordinate of this parabola at the point M.

Case II. Let $g > 2k$.

The general expression may be written

$$\Sigma = k\int \frac{d\mu}{\cos^2\mu \sqrt{\left\{1 - \left(\frac{g^2 - 4k^2}{g^2}\right)\sin^2\mu\right\}}} + k\int \frac{d\mu}{\cos^3\mu} \quad \ldots(12).$$

Let
$$\frac{g^2 - 4k^2}{g^2} = i^2 \quad \ldots\ldots\ldots\ldots\ldots(13),$$

and the last equation becomes

$$\Sigma = k\int \frac{d\mu}{\cos^2\mu \sqrt{(1 - i^2\sin^2\mu)}} + k\int \frac{d\mu}{\cos^3\mu} \quad \ldots\ldots(14).$$

Now U being the arc of an hyperbola, a the transverse axis, and $i^2 = \frac{a^2}{a^2 + b^2}$, we know that

$$\frac{iU}{a(1 - i^2)} = \int \frac{d\mu}{\cos^2\mu \sqrt{(1 - i^2\sin^2\mu)}} \quad \ldots\ldots(15);$$

hence if $k = \frac{a(1 - i^2)}{i}$, we shall have

Logarithmic parabola = plane hyperbola + plane parabola
$\ldots\ldots(16).$

The semiaxes a, b of this hyperbola may easily be determined by the equations

$$k = \frac{a(1 - i^2)}{i}, \quad i^2 = \frac{a^2}{a^2 + b^2}; \quad \text{or } a^2 = \frac{g^2(g^2 - 4k^2)}{16k^2}, \quad b = \frac{g}{2} \quad \ldots(17).$$

We may eliminate the arc of the hyperbola and introduce instead, elliptic integrals of the first and second orders.

Let $\sqrt{I} = 1 - i^2\sin^2\mu$, then

$$\frac{\Sigma}{k} = \int \frac{d\mu}{\cos^2\mu \sqrt{I}} + \int \frac{d\mu}{\cos^3\mu}$$

and the formula, for comparing elliptic integrals with reciprocal parameters, gives

$$\int \frac{d\mu}{\cos^2\mu \sqrt{I}} + \int \frac{d\mu}{I\sqrt{I}} = \int \frac{d\mu}{\sqrt{I}} + \frac{\tan\mu}{\sqrt{I}} \quad \ldots\ldots(18).$$

We have also

$$-\int \frac{d\mu}{I\sqrt{I}} = \frac{-1}{1 - i^2}\int d\mu \sqrt{I} + \frac{i^2}{1 - i^2}\,\frac{\sin\mu\cos\mu}{\sqrt{I}}.$$

Adding and reducing,

$$\Sigma = k\left[\int\frac{d\mu}{\sqrt{I}} + \int\frac{d\mu}{\cos^3\mu}\right] + \frac{g^2}{4k}\left[\tan\mu\sqrt{I} - \int d\mu\sqrt{I}\right] \ldots (19).$$

Case III. Let $2k > g$.

To integrate in this case, we must transform the second member of the equation (8). Assume

$$2k\tan\mu = g\tan\nu \ldots\ldots\ldots\ldots (20).$$

Then if we make $\frac{4k^2 - g^2}{4k^2} = j^2$, we shall have

$$2(2g + x) = g + g\sec\nu, \quad \text{and} \quad \frac{dx}{d\nu} = \frac{g}{2}\frac{\sin\nu}{\cos^2\nu}.$$

But $$\sin^2\mu = \frac{g^2\sin^2\nu}{4k^2(1 - j^2\sin^2\nu)},$$

hence $$\frac{d\Sigma'}{d\nu} = \frac{g}{2}\frac{\sqrt{(1 - j^2\sin^2\nu)}}{\cos^2\nu} + \frac{g}{2}\frac{\sqrt{(1 - j^2\sin^2\nu)}}{\cos^3\nu} \ldots (21).$$

Now since $$\frac{kd\mu}{\cos^2\mu} = \frac{g}{2}\frac{d\nu}{\cos^2\nu},$$

and $$\cos\mu = \frac{\cos\nu}{\sqrt{(1 - j^2\sin^2\nu)}} = \frac{\cos\nu}{\sqrt{J}}.$$

Writing J for $(1 - j^2\sin^2\nu)$, we shall have

$$k\int\frac{d\mu}{\cos^3\mu} = \frac{g}{2}\int\frac{d\nu\sqrt{(1 - j^2\sin^2\nu)}}{\cos^3\nu},$$

$$\text{or } \Sigma' = \frac{gj^2}{2}\int\frac{d\nu}{\sqrt{(1 - j^2\sin^2\nu)}} + \frac{g}{2}(1 - j^2)\int\frac{d\nu}{\cos^2\nu\sqrt{(1 - j^2\sin^2\nu)}} + k\int\frac{d\mu}{\cos^3\mu}$$
$$\ldots\ldots (22).$$

Now the second term of the right-hand member of this equation is the expression for an arc of a hyperbola, the distance between whose foci is g. Hence

$$\Sigma' = \frac{gj^2}{2}\int\frac{d\nu}{\sqrt{(1 - j^2\sin^2\nu)}} + U + \Pi \ldots\ldots (23),$$

Π being an arc of the parabola.

We may eliminate the function of the first order and represent in this case the arc of the logarithmic parabola by the arcs of an ellipse, an hyperbola, and a parabola.

Let U be the arc of an hyperbola whose semi-transverse axis is $\frac{1}{j}$, and putting E and Π for the elliptic and parabolic arcs,

$$\Sigma' = \frac{g}{2} \cdot \frac{j^2}{(1-j^2)} \left[U_{(\nu)} + E_j(\nu) - \tan \nu \sqrt{J} \right] + \Pi(\mu) \ldots (24);$$

or, as the equation may be written,

$$\Sigma' = \frac{g}{2} \left[\int \frac{d\nu}{\sqrt{J}} - \int d\nu \sqrt{J} + \tan \nu \sqrt{J} \right] + k \int \frac{d\mu}{\cos^3 \mu} \quad \ldots (25).$$

[*To be continued.*]

ON THE THEORY OF PERMUTANTS.

By Arthur Cayley.

A FORM may be considered as composed of blanks which are to be filled up by inserting in them specializing characters, and a form the blanks of which are so filled up becomes a symbol. We may for brevity speak of the blanks of a symbol in the sense of the blanks of the form from which such symbol is derived. Suppose the characters are 1, 2, 3, 4,... the symbol may always be represented in the first instance and without reference to the nature of the form, by $V_{1234}\ldots$ And it will be proper to consider the blanks as having an invariable order to which reference will implicitly be made; thus, in speaking of the characters 2, 1, 3, 4,.... instead of as before 1, 2, 4,.... the symbol will be $V_{2134}\ldots$ instead of $V_{1234}\ldots$ When the form is given we shall have an equation such as

$$V_{1234} = P_{12}Q_3R_4\ldots. \quad \text{or} = P_{12}P_{34}\ldots \text{ \&c.},$$

according to the particular nature of the form.

Consider now the characters 1, 2, 3, 4,.... and let the primitive arrangement and every arrangement derivable from it by means of an even number of inversions or interchanges of two characters be considered as positive, and the arrangements derived from the primitive arrangement by an odd number of inversions or interchanges of two characters be considered as negative; a rule which may be termed "the rule of signs." The aggregate of the symbols which correspond to every possible arrangement of the characters,

giving to each symbol the sign of the arrangement, may be termed a "Permutant;" or, in distinction from the more general functions which will presently be considered, a simple permutant, and may be represented by enclosing the symbol in brackets, thus $(V_{1234..})$. And by using an expression still more elliptical than the blanks of a symbol, we may speak of the blanks of a permutant, or the characters of a permutant.

As an instance of a simple permutant, we may take

$$(V_{123}) = V_{123} + V_{231} + V_{312} - V_{132} - V_{213} - V_{321}.$$

And if in particular $V_{123} = a_1b_2c_3$, then

$$(V_{123}) = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1.$$

It follows at once that a simple permutant remains unaltered, to the sign *près* according to the rule of signs, by any permutations of the characters entering into the permutant. For instance,

$$(V_{123}) = (V_{231}) = (V_{312}) = -(V_{132}) = -(V_{213}) = -(V_{321}).$$

Consequently also when two or more of the characters are identical, the permutant vanishes, thus

$$V_{113} = 0.$$

The form of the symbol may be such that the symbol remains unaltered, to the sign *près* according to the rule of signs, for any permutations of the characters in certain particular blanks. Such a system of blanks may be termed a quote. Thus, if the first and second blanks are a quote,

$$V_{123} = -V_{213}, \quad V_{132} = -V_{312}, \quad V_{231} = -V_{321},$$

and consequently

$$(V_{123}) = 2(V_{123} + V_{231} + V_{312}).$$

And if the blanks constitute one single quote,

$$(V_{123\cdots}) = NV_{123\cdots},$$

where $N = 1.2.3.\ldots n$, n being the number of characters. An important case, which will be noticed in the sequel, is that in which the whole series of blanks divide themselves into quotes, each of them containing the same number of blanks. Thus, if the first and second blanks, and the third and fourth blanks, form quotes respectively,

$$\tfrac{1}{6}(V_{1234}) = V_{1234} + V_{1342} + V_{1423} + V_{3412} + V_{4213} + V_{2314}.$$

It is easy now to pass to the general definition of a "Permutant." We have only to consider the blanks as forming

not as heretofore a single set, but any number of distinct sets, and to consider the characters in each set of blanks as permutable *inter se* and not otherwise, giving to the symbol the sign compounded of the signs corresponding to the arrangements of the characters in the different sets of blanks. Thus, if the first and second blanks form a set, and the third and fourth blanks form a set,

$$((V_{1234})) = V_{1234} - V_{2134} - V_{1243} + V_{2143}.$$

The word 'set' will be used throughout in the above technical sense. The particular mode in which the blanks are divided into sets may be indicated either in words or by some superadded notation. It is clear that the theory of permutants depends ultimately on that of simple permutants; for if in a compound permutant we first write down all the terms which can be obtained, leaving unpermuted the characters of a particular set, and replace each of the terms so obtained by a simple permutant having for its characters the characters of the previously unpermuted set, the result is obviously the original compound permutant. Thus, in the above-mentioned case, where the first and second blanks form a set and the third and fourth blanks form a set,

$$((V_{1234})) = (V_{1234}) - (V_{1243}),$$

or

$$((V_{1234})) = (V_{1234}) - (V_{2134}),$$

in the former of which equations the first and second blanks in each of the permutants on the second side form a set, and in the latter the third and fourth blanks in each of the permutants on the second side form a set, the remaining blanks being simply supernumerary and the characters in them unpermutable. It should be noted that the term quote, as previously defined, is only applicable to a system of blanks belonging to the same set, and it does not appear that anything would be gained by removing this restriction.

The following rule for the expansion of a simple permutant (and which may be at once extended to compound permutants) is obvious. Write down all the distinct terms that can be obtained, on the supposition that the blanks group themselves in any manner into quotes, and replace each of the terms so obtained by a compound permutant having for a distinct set the blanks of each assumed quote; the result is the original simple permutant. Thus in the simple permutant (V_{1234}), supposing for the moment that the first and second blanks form a quote, and that the third and fourth blanks form a quote, this leads to the equation

$$(V_{1234}) = +((V_{1234}))+((V_{1342}))+((V_{1423}))+((V_{3412}))+((V_{4213}))+((V_{2314})),$$

where in each of the permutants on the second side the first and second blanks form a set, and also the third and fourth blanks.

The blanks of a simple or compound permutant may of course, without either gain or loss of generality, be considered as having any particular arrangement in space, for instance, in the form of a rectangle: thus $V_{\substack{12\\34}}$ is neither more nor less general than V_{1234}. The idea of some such arrangement naturally presents itself as affording a means of shewing in what manner the blanks are grouped into sets. But, considering the blanks as so arranged in a rectangular form or in lines and columns, suppose in the first instance that this arrangement is independent of the grouping of the blanks into sets, or that the blanks of each set or of any of them are distributed at random in the different lines and columns. Assume that the form is such that a symbol

$$V_{\substack{\alpha\beta\gamma\ldots\\ \alpha'\beta'\gamma'\ldots\\ \vdots}}$$

is a function of symbols $V_{\alpha\beta\gamma\ldots}$, $V_{\alpha'\beta'\gamma'\ldots}$, &c. Or, passing over this general case and the case of intermediate generality of the function being a symmetrical function, assume that

$$V_{\substack{\alpha\beta\gamma\ldots\\ \alpha'\beta'\gamma'\ldots\\ \vdots}}$$

is the product of symbols $V_{\alpha\beta\gamma\ldots}$, $V_{\alpha'\beta'\gamma'\ldots}$, &c. Upon this assumption it becomes important to distinguish the different ways in which the blanks of a set are distributed in the different lines and columns. The cases to be considered are: (*A*). The blanks of a single set or of single sets are situated in more than one column. (*B*). The blanks of each single set are situated in the same column. (*C*). The blanks of each single set form a separate column. The case (*B*) (which includes the case (*C*)) and the case (*C*) merit particular consideration. In fact the case (*B*) is that of the functions which I have, in my memoir on Linear Transformations in the *Journal*, called hyperdeterminants, and the case (*C*) is that of the particular class of hyperdeterminants previously treated of by me in the *Cambridge Philosophical Transactions*, and also particularly noticed in the memoir on Linear Transformations. The functions of the case (*B*) I now propose to call "Intermutants," and those in the case (*C*) "Commutants." Commutants include

as a particular case "Determinants," which term will be used in its ordinary signification. The case (*A*) I shall not at present discuss in its generality, but only with the further assumption that the blanks form a single set (this, if nothing further were added, would render the arrangement of the blanks into lines and columns valueless), and moreover that the blanks of each line form a quote: the permutants of this class (from their connexion with the researches of Pfaff on differential equations) I shall term "Pfaffians." And first of commutants, which, as before remarked, include determinants.

The general expression of a commutant is

$$(V_{\substack{11\ldots \\ 22 \\ \vdots \\ nn}}). \qquad \text{Or} \quad \begin{pmatrix} 11\ldots \\ 22 \\ \vdots \\ nn \end{pmatrix}$$

and (stating again for this particular case the general rule for the formation of a permutant) if, permuting the characters in the same column in every possible way, considering these permutations as positive or negative according to the rule of signs, one system be represented by

$$\begin{matrix} r_1 s_1 \ldots \\ r_2 s_2 \\ \vdots \\ r_n s_n \end{matrix}$$

the commutant is the sum of all the different terms

$$\pm\pm\ldots V_{r_1 s_1} \ldots V_{r_2 s_2} \ldots V_{r_n s_n} \ldots$$

The different permutations may be formed as follows: first permute the characters in all the columns except a single column, and in each of the arrangements so obtained permute entire lines of characters. It is obvious that, considering any one of the arrangements obtained by permutations of the characters in all the columns but one, the permutations of entire lines and the addition of the proper sign will only reproduce the same symbol—in the case of an even number of columns constantly with the positive sign, but in the case of an odd number of columns with the positive or negative sign, according to the rule of signs. For the inversion or interchange of two entire lines is equivalent to as many inversions or interchanges of two characters as there are characters in a line, *i.e.* as there are columns, and consequently introduces a sign compounded of as many negative signs as there are columns. Hence

THEOREM. A commutant of an even number of columns may be calculated by considering the characters of any one column (no matter which) as supernumerary unpermutable characters, and multiplying the result by the number of permutations of as many things as there are lines in the commutant.

The mark † added to a commutant of an even number of columns will be employed to shew that the numerical multiplier is to be omitted. The same mark placed over any one of the columns of the commutant will shew that the characters of that particular column are considered as non-permutable.

A determinant is consequently represented indifferently by the notations

$$\begin{vmatrix} 11 \\ 22 \\ \vdots \\ nn \end{vmatrix}^{\dagger} \quad \begin{vmatrix} \overset{\dagger}{11} \\ 22 \\ \vdots \\ nn \end{vmatrix}, \quad \begin{vmatrix} \overset{\dagger}{11} \\ 22 \\ \vdots \\ nn \end{vmatrix}.$$

And a commutant of an odd number of columns vanishes identically.

By considering, however, a commutant of an odd number of columns, having the characters of some one column non-permutable, we obtain what will in the sequel be spoken of as commutants of an odd number of columns. This non-permutability will be denoted, as before, by means of the mark † placed over the column in question, and it is to be noticed that it is not, as in the case of a commutant of an even number of columns, indifferent over which of the columns the mark in question is placed; and consequently there would be no meaning in simply adding the mark † to a commutant of an odd number of columns.

A commutant is said to be symmetrical when the symbols $V_{\alpha\beta\gamma..}$ are such as to remain unaltered by any permutations *inter se* of the characters $\alpha, \beta, \gamma...$ A commutant is said to be skew when each symbol $V_{\alpha\beta\gamma..}$ is such as to be altered in sign only according to the rule of signs for any permutation *inter se* of the characters $\alpha, \beta, \gamma...$, which of course implies that the symbol $V_{\alpha\beta\gamma..}$ vanishes when any two of the characters $\alpha, \beta, \gamma...$ are identical. The commutant is said to be demiskew when $V_{\alpha,\beta,\gamma..}$ is altered in sign only according to the rule of signs for any permutation *inter se* of non-identical characters $\alpha, \beta, \gamma,...$

An intermutant is represented by a notation similar to that of a commutant. The sets are to be distinguished, whenever it is possible to do so, by placing in contiguity the symbols of the same set and separating them by a stroke or bar from the symbols of the adjacent sets. If, however, the symbols of the same set cannot be placed contiguously, we may distinguish the symbols of a set by annexing to them some auxiliary character by way of suffix or otherwise, these auxiliary symbols being omitted in the final result. Thus

$$\begin{pmatrix} 1 & 1 & 1a \\ \underline{2} & 2 & 2b \\ 3 & 3 & 5a \\ 4 & \overline{3} & 6b \end{pmatrix}$$

would shew that the 1, 2 of the first column and the 3, 4 of the same column, the 1, 2 and the upper 3 of the second column, the lower 3 of the same column, the 1, 5 of the third column, and the 2, 6 of the same column, form so many distinct sets,—the intermutant containing therefore

$$(2.2.6.1.2.2 =)\ 96 \text{ terms.}$$

A commutant of an even number of columns may be considered as an intermutant such that the characters of some one (no matter which) of its columns form each of them by itself a distinct set, and in like manner a commutant of an odd number of columns may be considered as an intermutant such that the characters of some one determinate column form each of them by itself a distinct set.

The distinction of symmetrical, skew and demiskew applies obviously as well to intermutants as to commutants. The theory of skew intermutants and skew commutants has a connexion with that of Pfaffians.

Suppose $V_{\alpha,\beta,\gamma..} = V_{\alpha+\beta+\gamma..}$ (which implies the symmetry of the intermutant or commutant) and write for shortness $V_0 = a$, $V_1 = b$, $V_2 = c$, &c. Then

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = 2(ac - b^2),$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = 2(ae - 4bd + 3c^2),$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}^{\dagger} = (ac - b^2), \text{ &c.}$$

The functions on the second side are evidently hyperdeterminants such as are discussed in my memoir on Linear

Transformations, and there is no difficulty in forming directly from the intermutant or commutant on the first side of the equation the symbol of derivation (in the sense of the memoir on Linear Transformations) from which the hyperdeterminant is obtained. Thus

$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ is $\overline{12}^2.UU$, $\quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ is $\overline{12}^4.UU$,

$\begin{bmatrix} \overset{\dagger}{0} & 0 \\ 1 & 1 \end{bmatrix}$ is $\overline{12}\,U^0U^1$, $\quad \begin{bmatrix} 0 & 0 & \overset{\dagger}{0} & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ is $\overline{12}^3U^0U^1$,

each permutable column $\begin{matrix} 0 \\ 1 \end{matrix}$ corresponding to a $\overline{12}$* and a non-permutable column $\begin{matrix} \overset{\dagger}{0} \\ 1 \end{matrix}$ changing UU into U^0U^1. Similarly

$\begin{Vmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 2 \end{Vmatrix}$ becomes $(\overline{12}\,.\,\overline{13}\,.\,\overline{23})^2.UUU$,

$\begin{Vmatrix} \overset{\dagger}{0} & 0 \\ 1 & 1 \\ 2 & 2 \end{Vmatrix}$ becomes $\overline{12}\,.\overline{13}\,.\overline{23}\;U^0U^1U^2$,

$\begin{Vmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{Vmatrix}$ becomes $(\overline{12}.\overline{13}.\overline{14}.\overline{23}.\overline{24}.\overline{34})^2UUUU$, &c.

The analogy would be closer if in the memoir on Linear Transformations, just as $\overline{12}$ is used to signify $\begin{vmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{vmatrix}$, $\overline{123}$ had been used to signify $\begin{vmatrix} \xi_1^2 & \xi_1\eta_1 & \eta_1^2 \\ \xi_2^2 & \xi_2\eta_2 & \eta_2^2 \\ \xi_3^2 & \xi_3\eta_3 & \eta_3^2 \end{vmatrix}$ &c., for then $\begin{Vmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 2 \end{Vmatrix}$ would have corresponded to $\overline{123}^2.UUU$, $\begin{Vmatrix} \overset{\dagger}{0} & 0 \\ 1 & 1 \\ 2 & 2 \end{Vmatrix}$ to $\overline{123}\;U^0U^1U^2$;

* Viz. $\begin{matrix} 0 \\ 1 \end{matrix}$ corresponds to $\overline{12}$ because 0 and 1 are the characters occupying the *first* and *second* blanks of a column. If 0 and 1 had been the characters occupying the *second* and *third* blanks in a column, the symbol would have been $\overline{23}$ and so on. It will be remembered, that the symbolic numbers 1, 2.... in the hyperdeterminant notation are merely introduced to distinguish from each other functions which are made identical after certain differentiations are performed.

and this would not only have been an addition of some importance to the theory, but would in some instances have facilitated the calculation of hyperdeterminants. The preceding remarks shew that the intermutant $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & \overline{1} \\ \overline{0} & \overline{0} & 0 \\ 1 & 1 & \overline{1} \end{pmatrix}$ (where the first and fourth blanks in the last column are to be considered as belonging to the same set) is in the hyperdeterminant notation $(12.34)^2.(14.23)UUUU$.

It will, I think, illustrate the general theory to perform the development of the last-mentioned intermutant. We have

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & \overline{1} \\ \overline{0} & \overline{0} & 0 \\ 1 & 1 & \overline{1} \end{pmatrix} = \overset{\dagger}{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \overline{0} & \overline{0} & 0 \\ 1 & 1 & 1 \end{pmatrix}} - \overset{\dagger}{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \overline{0} & \overline{0} & 1 \\ 1 & 1 & 1 \end{pmatrix}} - \overset{\dagger}{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ \overline{0} & \overline{0} & 0 \\ 1 & 1 & 0 \end{pmatrix}} + \overset{\dagger}{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ \overline{0} & \overline{0} & 1 \\ 1 & 1 & 0 \end{pmatrix}}$$

$$= 2 \left\{ \overset{\dagger}{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}} \overset{\dagger}{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}} - \overset{\dagger}{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}} \overset{\dagger}{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}} \right\}$$

$$= 2 \{(ad - bc)^2 - 4(ac - b^2)(bd - c^2)\},$$

$$= 2 (a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd),$$

the different steps of which may be easily verified.

The following important theorem (which is, I believe, the same as a theorem of Mr. Sylvester's, published in the *Philosophical Magazine*) is perhaps best exhibited by means of a simple example. Consider the intermutant

$$\begin{pmatrix} x & 1 \\ \underline{y} & 4 \\ x & 3 \\ y & 2 \end{pmatrix}$$

where in the first column the sets are distinguished as before by the horizontal bar, but in the second column the 1, 2 are to be considered as forming a set, and the 3, 4 as forming a second set. Then, partially expanding, the intermutant is

$$\overset{\dagger}{\begin{pmatrix} x & 1 \\ y & 4 \\ x & 3 \\ y & 2 \end{pmatrix}} - \overset{\dagger}{\begin{pmatrix} y & 1 \\ x & 4 \\ x & 3 \\ y & 2 \end{pmatrix}} - \overset{\dagger}{\begin{pmatrix} x & 1 \\ y & 4 \\ y & 3 \\ x & 2 \end{pmatrix}} + \overset{\dagger}{\begin{pmatrix} y & 1 \\ x & 4 \\ y & 3 \\ x & 2 \end{pmatrix}}$$

Or, since entire horizontal lines may obviously be permuted,

$$\begin{vmatrix} x & 1 \\ y & 2 \\ x & 3 \\ y & 4 \end{vmatrix}^{\dagger} - \begin{vmatrix} y & 1 \\ y & 2 \\ x & 3 \\ x & 4 \end{vmatrix}^{\dagger} - \begin{vmatrix} x & 1 \\ x & 2 \\ y & 3 \\ y & 4 \end{vmatrix}^{\dagger} + \begin{vmatrix} y & 1 \\ x & 2 \\ y & 3 \\ x & 4 \end{vmatrix}^{\dagger}$$

And, observing that the 1, 2 form a permutable system as do also the 3, 4, the second and third terms vanish, while the first and fourth terms are equivalent to each other; we may therefore write

$$2 \begin{vmatrix} x & 1 \\ \underline{y} & \underline{2} \\ \overline{x} & \overline{3} \\ y & 4 \end{vmatrix}^{\dagger} = \begin{vmatrix} x & 1 \\ y & 4 \\ \overline{x} & \overline{3} \\ y & 2 \end{vmatrix}$$

where on the first side of the equation the bar has been introduced into the second column, in order to shew that *throughout* the equation the 1, 2 and the 3, 4 are to be considered as forming distinct sets.

Consider in like manner the expression

$$\begin{vmatrix} x & 1 \\ y & 7 \\ \underline{z} & \underline{6} \\ \overline{x} & \overline{8} \\ y & 2 \\ \underline{z} & \underline{9} \\ \overline{x} & \overline{4} \\ y & 5 \\ z & 3 \end{vmatrix}$$

where in the first column the sets are distinguished by the horizontal bars and in the second column the characters 1, 2, 3 and 4, 5, 6 and 7, 8, 9 are to be considered as belonging to distinct sets. The same reasoning as in the former case will shew that this is a multiple of

$$\begin{vmatrix} x & 1 \\ y & 2 \\ \underline{z} & \underline{3} \\ \overline{x} & \overline{4} \\ y & 5 \\ \underline{z} & \underline{6} \\ \overline{x} & \overline{7} \\ y & 8 \\ z & 9 \end{vmatrix}$$

And to find the numerical multiplier it is only necessary to inquire in how many ways, in the expression first written down, the characters of the first column can be permuted so that x, y, z may go with 1, 2, 3 and with 4, 5, 6 and with 7, 8, 9. The order of the x, y, z in the second triad may be considered as arbitrary; but once assumed, it determines the place of one of the letters in the first triad; for instance, $x8$ and $z9$ determine $y7$. The first triad must therefore contain $x1$ and $z6$ or $x6$ and $z1$. Suppose the former, then the third triad must contain $z3$, but the remaining two combinations may be either $x4, y5$, or $x5, y4$. Similarly, if the first triad contained $x6, z1$, there would be two forms of the third triad, or a given form of the second triad gives four different forms. There are therefore in all 24 forms, or

$$24 \begin{vmatrix} x\ 1 \\ y\ 2 \\ z\ 3 \\ \bar{x}\ 4 \\ y\ 5 \\ z\ 6 \\ \bar{x}\ 7 \\ y\ 8 \\ z\ 9 \end{vmatrix}^{\dagger} = \begin{vmatrix} x\ 1 \\ y\ 7 \\ z\ 6 \\ \bar{x}\ 8 \\ y\ 2 \\ z\ 9 \\ \bar{x}\ 4 \\ y\ 5 \\ z\ 3 \end{vmatrix}$$

where the bars in the second column on the first side shew that *throughout* the equation 1, 2, 3 and 4, 5, 6 and 7, 8, 9 are to be considered as forming distinct sets. The above proof is in reality perfectly general, and it seems hardly necessary to render it so in terms.

To perceive the significance of the above equation it should be noticed that the first side is a product of determinants, viz.

$$24 \begin{vmatrix} x\ 1 \\ y\ 2 \\ z\ 3 \end{vmatrix}^{\dagger} \begin{vmatrix} x\ 5 \\ y\ 6 \\ z\ 7 \end{vmatrix}^{\dagger} \begin{vmatrix} x\ 7 \\ y\ 8 \\ z\ 9 \end{vmatrix}^{\dagger}$$

And if the second side be partially expanded by permuting the characters of the second column, each of the terms so obtained is in like manner a product of determinants, so that

$$24 \begin{vmatrix} x\ 1 \\ y\ 2 \\ z\ 3 \end{vmatrix}^{\dagger} \begin{vmatrix} x\ 4 \\ y\ 5 \\ z\ 6 \end{vmatrix}^{\dagger} \begin{vmatrix} x\ 7 \\ y\ 8 \\ z\ 9 \end{vmatrix}^{\dagger} = \begin{vmatrix} x\ 1 \\ y\ 7 \\ z\ 6 \end{vmatrix}^{\dagger} \begin{vmatrix} x\ 8 \\ y\ 2 \\ z\ 9 \end{vmatrix}^{\dagger} \begin{vmatrix} x\ 4 \\ y\ 5 \\ z\ 3 \end{vmatrix}^{\dagger} \pm \&c.,$$

the permutations on the second side being the permutations *inter se* of 1, 2, 3, of 4, 5, 6, and of 7, 8, 9.

It is obvious that the preceding theorem is not confined to intermutants of two columns.

[*To be continued.*]

POSTSCRIPT.

I wish to explain as accurately as I am able, the extent of my obligations to Mr. Sylvester in respect of the subject of the present memoir. The term permutant is due to him—intermutant and commutant are merely terms framed between us in analogy with permutant, and the names date from the present year (1851). The theory of commutants is given in my memoir in the *Cambridge Philosophical Transactions*, and is presupposed in the memoir on Linear Transformations. It will appear by the last-mentioned memoir that it was by representing the coefficients of a biquadratic function by $a = 1111$, $b = 1112 = 1121 = \&c.$, $c = 1122 = \&c.$, $d = 1222 = \&c.$, $e = 2222$, and forming the commutant $\begin{Vmatrix} 1111 \\ 2222 \end{Vmatrix}$ that I was led to the function $ae - 4bd + 3c^2$. The function $ace + 2bcd - ad^2 - b^2e - c^3$ or $\begin{vmatrix} a, b, c \\ b, c, d \\ c, d, e \end{vmatrix}$ is mentioned in the memoir on Linear Transformations, as brought into notice by Mr. Boole. From the particular mode in which the coefficients a, b,... were represented by symbols such as 1111, &c., I did not perceive that the last-mentioned function could be expressed in the commutant notation. The notion of a permutant, in its most general sense, is explained by me in my memoir (*Crelle*, tom. XXXVIII. p. 93) "Sur les determinants gauches." See the paragraph (p. 94) commencing "On obtient ces fonctions, &c." and which should run as follows: "On obtient ces fonctions (dont je reprends ici la théorie) par les propriétés génerales d'un determinant défini comme suit. En exprimant &c.;" the sentence as printed being "..... défini. Car en exprimant &c." which confuses the sense. Some time in the present year (1851) Mr. Sylvester, in conversation, made to me the very important remark, that as one of a class the above-mentioned function, $ace + 2bcd - ad^2 - b^2e - c^3$, could be expressed in the commutant notation $\begin{Vmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 2 \end{Vmatrix}$, viz. by considering $00 = a$, $01 = 10 = b$, $02 = 11 = 20 = c$, $12 = 21 = d$, $22 = e$; and the subject being thereby recalled to my notice, I found shortly afterwards the expression for the function

$$a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd$$

(which cannot be expressed as a commutant) in the form of an intermutant, and I was thence led to see the identity, so to say, of the theory of hyperdeterminants, as given in the memoir on Linear Transformations, with the present theory of intermutants. It is understood between Mr. Sylvester and myself, that the publication of the present memoir is not to affect Mr. Sylvester's right to claim the origination, and to be considered as the first publisher of such part as may belong to him of the theory here sketched out.

ON THE PRINCIPLES OF THE CALCULUS OF FORMS.

By J. J. Sylvester, M.A., F.R.S.

PART I.—GENERATION OF FORMS.*

Sect. I.—*On Simple Concomitance.*

The primary object of the Calculus of Forms is the determination of the properties of Rational Integral Homogeneous Functions or systems of functions: this is effected by means of transformation; but to effect such transformation experience has shewn that forms or form-systems must be contemplated not merely as they are in themselves, but with reference to the ensemble of forms capable of being derived from them, and which constitute as it were an unseen atmosphere around them. The first part of this essay will therefore be devoted to the theory of the external relations of forms or form-systems; the second part to the analysis of forms: that is to say, the first part will treat of the Generation and affinities, and the second part of the Reduction and equivalences of forms.

In its most crude and absolute, or, so to speak, archetypal condition a Rational Integral Homogeneous Function may be regarded as a linear function of several distinct and perfectly independent classes of variables. The first step towards the limitation of this very general but necessary conception consists in imagining the total number of classes to become segregated into groups, and certain correspondencies to obtain between the variables of a class in any group with some the variables in each other class of the same group. The investigations in this and the subsequent section will be confined exclusively to the theory of functions where the several classes of variables, if more than one, all belong to a single group, so that the variables in one class have each their respective correspon-

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* It may be well at the outset to give notice to my readers of the exact meaning to be attached to the following terms:

1. The linear-transformations are supposed to be always taken such that the modulus, *i.e.* the determinant of the coefficients of transformation, is unity; or, as it may be phrased, the transformations are uni-modular.

2. The word Determinant is restricted in all cases to signify the alternate function formed in the usual manner from a group of quantities arranged in square order.

3. The word *Discriminant* (typified by the prefix-symbol □) is used to denote the determinant (usually but most perplexingly so called) of a homogeneous function of variables.

4. The resultant of two or more homogeneous functions of as many variables is the left-hand side of the final equation (in its *complete form* and free from extraneous factors) which results from eliminating the variables between the equations obtained by making each of the functions zero.

dents in the remaining classes. Such a group may again be conceived to become subdivided into sets each of the same number of variables, and the corresponding variables in the different sets to become absolutely identical. This leads to the conception of a homogeneous function of related classes of variables of various degrees of exponency in respect to the several classes. The relation of the different classes, if containing the same number of variables (in which case the relation may be termed Simple) will be understood to be defined by their being simultaneously subject to similar or contrary operations of linear substitution; so that, for example, if x, y, z; ξ, η, ζ are two such classes when x, y, z are replaced by $ax + by + cz$, $a'x + b'y + c'z$, $a''x + b''y + c''z$, respectively, ξ, η, ζ will be, according to the *species* of the relation, subject to be at the same time replaced either by $a\xi + b\eta + c\zeta$, $a'\xi + b'\eta + c'\zeta$, $a''\xi + b''\eta + c''\zeta$, or otherwise by $a\xi + \beta\eta + \gamma\zeta$, $a'\xi + \beta'\eta + \gamma'\zeta$, $a''\xi + \beta''\eta + \gamma''\zeta$, where

$$a = \begin{matrix} 1 & 0 & 0, \\ 0 & b' & c', \\ 0 & b'' & c'', \end{matrix} \qquad \beta = \begin{matrix} 0 & 1 & 0, \\ a' & 0 & c', \\ a & 0 & c'', \end{matrix} \qquad \gamma = \begin{matrix} 0 & 0 & 1, \\ a' & b' & 0, \\ a'' & b'' & 0, \end{matrix}$$

&c. &c. &c.*

On the former supposition the related classes $x, y, z, \xi, \eta, \zeta$ will be said to be cogredient, and on the latter supposition contragredient.† If now we have one or more functions of classes of variables so related,‡ such function or system of functions may have associated with it a concomitant, also made up of distinct but related classes of variables, such classes being capable of being either greater or fewer in number than the classes of the given function or system of functions.

In the primitive function or system, as also in the concomitant, the related classes may be all of the same species, or some of one and the others of the contrary species. Even if we limit ourselves to the conception of a primitive function or system of functions with only one class of variables, its concomitant may be composed of various classes of variables, in respect to some of which it will be covariant with, and in respect to the others contravariant to, the primitive function

* See my paper in the preceding number of this *Journal*.

† The germ of the notion of contragredience will be found in the immortal *Arithmetic* of the great and venerable Gauss.

‡ The relation here spoken of will be observed to be of a *dynamical* character, not referring to the systems as they are in themselves, but to the movements to which they are simultaneously subject.

or system.* This is an immense and most important extension of the conception of a concomitant given in my preceding paper in this *Journal*, and will be shewn to have the effect of reducing the whole existing theory under subjection to certain simple abstract and universal laws of operation.

The relation of concomitance is purely of form. A being a given form, B is its concomitant, when A' being derived from A by simultaneous substitutions impressed upon the class of variables or upon each of the classes (if there be more than one) in A, and B' from B by corresponding (coincident or contrary) substitutions impressed upon the class or classes of variables in B, B' is capable of being derived from A' after the same law as B from A; or, as it may be otherwise expressed, "functions are concomitant when their correlated linear derivatives are homogeneous in point of form."†

This definition implies that one at least of the forms must be the most general possible of its kind: in a secondary but very important sense, however, functions obtained by impressing particular values or relations upon the quantities entering into the primitive and its associate form, will still be called concomitant. Thus $x^3 - y^3$ will be termed a concomitant to $x^3 + y^3$, not that we can affirm that $(ax + by)^3 - (cx + dy)^3$:

i.e. $(a^3 - c^3)x^3 + 3(a^2b - c^2d)x^2y + 3(ab^2 - cd^2)z^2 + (b^3 - d^3)y^3$,

treated as a function of x and y, can be derived from $(ax + by)^3 + (cx + dy)^3$,

i.e. $(a^3 + c^3)x^3 + 3(a^2b + c^2d)x^2y + 3(ab^2 + cd^2)xy^2 + (b^3 + d^3)y^3$,

when $ad - bc = 1$ by the same law as $(x^3 - y^3)$ from $(x^3 + y^3)$, for the elements for forming such comparison are wanting, but because $x^3 + y^3$ and $x^3 - y^3$ are the correspondent particular values respectively assumed by $fx^3 + 3gx^2y + 3hxy^2 + ky^3$, and its concomitant

$$(ad^2 - 2c^3 - 3bcd)x^3 + (6b^2d - 3c^2b - 3acd)x^2y$$
$$+ (6ac^2 - 3cb^2 - 3cba)xy^2 + (a^2d + 2b^3 - 3bca)y^3,$$

when $$a = 1,\ b = 0,\ c = 0,\ d = 1.$$

With the aid of this extended signification of the term concomitant (whether it be a covariant or contravariant) we can in all cases speak (as otherwise we in general could not) of

* And of course the concomitant may be an invariant to its originant in respect of one or more systems of variables entering into the former.

† Or, more generally, it may be said that concomitance consists in the persistence of morphological affinity.

the concomitant of a concomitant. The relation between systems of variables has been stated to be Simple (whether they be congredient or contragredient) when each variable in one system corresponds with some one in each other. Compound relation arises as follows:—Suppose $x, y; \xi, \eta$ two independent systems of two variables each, and that the system of four variables u, v, w, t is subject to linear variations imitating, in the way of congredience or contragredience, those to which $x\xi, x\eta, y\xi, y\eta$ are subject; then u, v, w, t may be said to be congredient or contragredient to the continued systems $x, y; \xi, \eta$. If $x, y; \xi, \eta$ be themselves cogredient, then a system of only three variables u, v, w, may be cogredient or contragredient in respect to $x\xi, x\eta + y\xi, y\eta$, and if $x, y; \xi, \eta$ be coincident, u, v, w may be similarly related to x^2, xy, y^2. The illustration may easily be generalized, and it will be seen in the sequel that this conception of compound-relation between systems of a differing number of variables will greatly extend the power and application of the methods about to be developed. Without having recourse to a formal definition, it is obvious that the notion of a concomitant conveyed in my former paper in this *Journal* lends itself without difficulty to the most general supposition which can be made of functions between which any number of systems of related variables are distributed, whatever such relation be, whether simple or compound, and whether of cogredience or of contragredience. The proposition stated in my last paper relative to a concomitant of the concomitant of a function being a concomitant of the original still applies to concomitants in the wider sense in which we now understand that term, and the species of each system of variables in the second concomitant with respect to the species or either species (if there be systems of both kinds in the primitive) will be determined upon the general principle which determines the effect of concurrence and contrariety being made to operate each upon itself or one in either order upon the other.

The highest law and the most powerful in its applications which I have yet discovered in the theory of concomitants may be expressed by affirming that when several related classes of variables are present in any concomitant, a new concomitant, *derived from the former by treating one or any number of these classes as independent of the remaining classes*, will still be a concomitant of the primitive. I shall quote this hereafter as the Law of Succession. This law, to which I have been led up inductively, requires an extended examination

and a rigorous proof. It is the keystone of the subject, and any one who should suppose that it is a self-evident proposition (as from the simplicity of the enunciation it might be supposed to be) will commit no slight error.

If $\phi(x, y \ldots . z)$ be any homogeneous form of function of $x, y \ldots z$, every homogeneous sum in the expansion by Taylor's theorem of $\phi(u+u', v+v' \ldots w+w')$, which in fact, on making $u' = x, v' = y \ldots w' = z$, becomes identical (to a numerical factor près) with $\left(u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz}\right)^{i}\phi$, is what I have elsewhere termed an Emanant, and by a partial method I had demonstrated that every invariant of such an emanant in respect to $u, v \ldots w$, in which $x, y \ldots z$ are treated as constants, or *vice versâ*, would give a covariant of ϕ. The reason of this is now apparent. For it may easily be shewn* that every emanant is in fact itself a covariant of the function to which it belongs with respect to each of the related classes of variables which enter into it, or is as it may be termed a double covariant. The law of Succession shews therefore that a concomitant to an emanant from which one of the classes has disappeared will be a covariant of the primitive in respect to the remaining class.

In applying the law of Succession, great use can be made of a function of two classes of letters which may be termed a Universal Mixed Concomitant; this is $x\xi + y\eta + \ldots + z\zeta$, which has the property of remaining unaltered when any linear substitution (for which the modulus is unity) is impressed upon $x, y \ldots z$, and the contrary one upon $\xi, \eta \ldots \zeta$.†

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* To demonstrate this it is only necessary to observe that if $u, v, \ldots w$, $u', v', \ldots w'$ be cogredient with themselves and with $x, y, \ldots z$,

$$\phi(u + \lambda u', v + \lambda v', \ldots w + \lambda w')$$

will evidently be a concomitant of $\phi(x, y, \ldots z)$; and, λ being arbitrary, the coefficients of the different powers of λ must be separately concomitants of $\phi(x, y, \ldots z)$, but these coefficients are the emanants of ϕ. Q. E. D.

† Thus, if

$$\begin{aligned} x &= ax' + by' + cz', & \xi &= (gn - hm)\,\xi' + (hl - fn)\,\eta' + (fm - gl)\,\zeta', \\ y &= fx' + gy' + hz', & \eta &= (-nb + mc)\,\xi' + (-lc + na)\,\eta' + (-ma + lb)\,\zeta', \\ z &= lx' + my' + nz', & \zeta &= (bh - cg)\,\xi' + (cf - ah)\,\eta' + (ag - bf)\,\zeta', \end{aligned}$$

then

$$x\xi + y\eta + z\zeta = \begin{pmatrix} a & b & c \\ f & g & h \\ l & m & n \end{pmatrix} \times (x'\xi' + y'\eta' + z'\zeta')$$

$$= x'\xi' + y'\eta' + z'\zeta' :$$

when the coefficients of transformation correspond to the direction-cosines between one system of rectangular axes and another, the reciproeal system

Let $f(x, y)$ be any function of x, y, of the degree m, $f + \lambda(x\xi + y\eta)^m$ will be a mixed concomitant of f, it being evident that every function of concomitants of a function is itself a concomitant of the same.

Suppose now

$$f = ax^m + mbx^{m-1}y + m \cdot \frac{m-1}{2} c.x^{m-2}.y^2 + \&c.,$$

the concomitant becomes

$$(a + \lambda\xi^m)x^m + m(b + \lambda\xi^{m-1}\eta)x^{m-1}y + m\frac{m-1}{2}(c + \lambda\xi^{m-2}y^2) + \&c.$$

Consequently if P be any concomitant of f, P' obtained from P by writing $a + \lambda\xi^m$, $b + \lambda\xi^{m-1}\eta$, &c. for a, b, &c., will still be a concomitant of f; and by Taylor's theorem P' evidently equals

$$P + \left(\xi^m \frac{d}{da} + \xi^{m-1}\eta \frac{d}{db} + \&c.\right) P$$
$$+ \frac{1}{1.2}\left(\xi^m \frac{d}{da} + \xi^{m-1}\eta \frac{d}{dc} + \&c.\right)^2 P$$
$$+ \&c.$$

If we take P an invariant of f, we have M. Hermite's theorem* for $f(x, y)$, and precisely the same demonstration applies to the general case of $f(x, y \ldots z)$. P' is, by virtue of the general rule, a contravariant of f in respect to $\xi, \eta \ldots \zeta$: if P be taken a function containing one single system, and is

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is identical with the direct system; so that x, y, z; ξ, η, ζ, on this particular supposition, may be regarded indifferently as contragredient or as cogredient; accordingly they may be made identical, and then $x^2 + y^2 + z^2$ remains invariable, which is the well-known characteristic of orthogonal transformation. It may be observed here that there exists a special theory of concomitance limited to such species of linear transformations, which may be termed Conditional Concomitance, and I have found in several cases that the invariants of conditional concomitants turn out to be absolute invariants of the primitive. Much more important is the remark that there exists a theory of universal concomitants for an indefinite number instead of merely two systems of variables, as used in the text. In the sequel it will be seen that the application of this universal concomitant (like the touch of an enchanter's wand) serves to transmute covariants into contravariants, and back again, and causes single invariants to germinate and fructify into complete connected systems of forms.

* This theorem was first stated to me by Mr. Cayley, who, I understand, derived it from M. Eisenstein, under the form of a theorem of covariants, which of course it becomes on interchanging x, y with $-y, x$. But as a theorem of covariants it could not be extended to functions of more than two variables. M. Hermite appears to have discovered this theorem, under its more eligible form, subsequently to, but independently of M. Eisenstein.

also a contravariant to f in respect to that system, P' will be a double contravariant; and if we make the two systems in P' identical, we have the extension of M. Hermite's theorem alluded to by me in one of the notes to my last paper, wherein I have stated that "I may be taken any *covariant* of the function:" as regards the purpose of that statement, the word covariant was used in error for contravariant.

The preceding method may be viewed as a particular application of the general principle, that if $U_1, U_2 \ldots U_m$ be any m functions (whether concomitants any of them of the others or not), then any concomitant of $\lambda_1 . U_1 + \lambda_2 . U_2 + \ldots + \lambda_m . U_m$ being expressed as a function of $\lambda_1, \lambda_2 \ldots \lambda_m$, every coefficient in such expression will be a concomitant of the system $U_1, U_2 \ldots\ldots U_m$. Thus, for example, if U and V be two quadratic functions of n variables $x, y \ldots z$, the *discriminant* $\square(\lambda U + \mu V)$ will contain $n+1$ terms, of which the coefficients of the first and last will be $\square U$ and $\square V$; and every one of the $(n+1)$ coefficients will be a concomitant (of course an invariant) of U and V. These $(n+1)$ invariants will in fact constitute the fundamental scale of invariants to the system U and V, and every other invariant of U and V will be an explicit rational function of the $(n+1)$ terms of the scale. In connexion with this principle may be stated another relative to any system of homogeneous functions of a greater number of variables of the same class, viz. that if any set of the variables one less in number than the number of the functions be selected at will, and any invariant of a given kind be taken of the resultant of the functions in respect to the variables selected, all such invariants so formed will have an integral factor in common, and this common factor will be an invariant of the given system of functions.

It will be convenient to speak hereafter of systems for which the march of the linear substitutions is coincident as congredient, and those for which the march is contrary as contragredient systems.

Suppose m congredient classes of m variables, the determinant formed by writing the $m \times m$ quantities in square order will evidently be a universal covariant. Thus, take the two systems x, y; ξ, η. $x\eta - y\xi$ is an universal covariant, and evidently therefore F, which I use to denote $\phi(x, y) \times \phi(\xi, \eta) + \lambda(x\eta - y\xi)^m$, will be a covariant to $\phi(x, y)$. Let $\phi(x, y)$ be of m dimensions, any invariant of F will be an invariant of ϕ: thus, let the two systems x, y; ξ, η be treated as perfectly independent, and take the discriminant of F (viewed as a function of x, y; ξ, η), *i.e.* the resultant of

the four functions $\frac{dF}{dx}, \frac{dF}{dy}, \frac{dF}{d\xi}, \frac{dF}{d\eta}$; this resultant will be an invariant of ϕ; and λ being arbitrary, all the coefficients of its different powers will be invariants of ϕ. We thus fall upon another theorem of M. Hermite, viz. that if $\lambda = \frac{\phi(x, y) \times \phi(\xi, \eta)}{(x\xi - y\eta)^m}$, the coefficients of the equation which will give the minimum values of λ are invariants of ϕ. So more generally, any invariant of $f(x, y, \xi, \eta) - \lambda(x\xi - y\eta)^m$, f being of the degree m in x, y and in ξ, η, will be an invariant of f; and among other invariants may be taken the discriminant obtained by treating x, ξ, y, η as absolutely unrelated.

If f be a function of various classes each containing n covariables, and if not less than n of these classes be covariable classes, and after, by selecting at will any n of such systems as $x_1, y_1 \ldots z_1$; $x_2, y_2 \ldots z_2$; $\ldots\ldots x_n, y_n \ldots z_n$; the symbolical determinant

$$\begin{matrix} \frac{d}{dx_1}, & \frac{d}{dy_1} \ldots\ldots & \frac{d}{dz_1}, \\ \frac{d}{dx_2}, & \frac{d}{dy_2} \ldots\ldots & \frac{d}{dz_2}, \\ \ldots\ldots\ldots\ldots\ldots & & \\ \ldots\ldots\ldots\ldots\ldots & & \\ \frac{d}{dx_n}, & \frac{d}{dy_n} \ldots\ldots & \frac{d}{dz_n}, \end{matrix}$$

be expanded and written equal to D, $D^{\iota}.f$ will be a concomitant of f; and, more generally, by selecting different combinations of the covariable systems n and n together in every way possible, and forming corresponding symbols of operation $E, F \ldots H$, we shall have $D^{\iota}.E^{\iota'} \ldots H^{(\iota)}.f$ for all values of $\iota, \iota' \ldots\ldots (\iota)$, a covariant of f in respect to the classes so combined. This explains and contains the whole pith and marrow of Mr. Cayley's simple but admirable method of obtaining covariants and invariants (or, as termed by their author, hyperdeterminants) to a function ϕ_1 of a single system $x_1, y_1 \ldots\ldots z_1$; he forms similar functions $\phi_2 \ldots\ldots \phi_\mu$ of $x_2, y_2 \ldots z_2$; $\ldots x_\mu, y_\mu \ldots z_\mu$, and uses the product $\phi_1 \times \phi_2 \ldots \times \phi_\mu$ as a function f of μ systems: the multiple covariant obtained by operating thereupon becomes a simple covariant on identifying the different classes of covariables introduced in the procedure.

SECTION II.—*On Complex Concomitance.*

We have hitherto been engaged in considering only a particular case of concomitance, the true idea of which relates not to an individual associated form (as such), but to a complex of forms capable of degenerating into an individual form. Such a complex may be called a Plexus. A plexus of forms is concomitant to a given form or combination of forms under the following circumstances.

If (O) be the originant, meaning thereby the primitive form or system of forms, and P the concomitant plexus made up of the μ forms $P_1, P_2 \ldots {}'P_\mu$, and if, when by duly related linear substitutions, O becomes O', the plexus P becomes P', made up of the forms $P_1, P_2 \ldots P'_\mu$, and if the plexus $'P$ formed from O' after the same law as P from O be made up of the forms $'P_1, 'P_2 \ldots P_\mu$, then will each form in either of the plexuses P, P' be a linear function of all the forms in the other plexus, and the connecting constants in every such linear function will be functions of the coefficients of the substitution whereby O and P have become transformed into O' and P'.

A function forming part of a concomitant plexus may be termed a concomitantive. Concomitantives therefore usually have a joint relation to a common plexus and a concomitant is only another name for an unique concomitantive. Every plexus contains a definite number of concomitantives; in place of any one of these may be substituted an arbitrary linear function of all the rest, but the total number of independent forms *sufficient and necessary* to make the complete plexus respond to the requirements of the definition will remain constant.

If now we combine together the whole number of functions contained in one or more plexuses concomitant to any given originant, all of the same degree relative to any given selected system or systems of variables, and if the number of the concomitantives so combined be exactly equal to the number of terms in each, arranged as a function of the selected class or classes of variables, then the dialytic resultant (obtained by treating each combination of the selected variables as an independent variable, and forming a determinant in the usual manner), will be a concomitant to the given originant. This, which is only the partial expansion of some much higher law, may be termed the "Law of Synthesis."

Let f be any function of a single class of variables $x_1, x_2 \ldots x_n$. Let χ represent any product of these variables or of their

several powers of any given degree r; the number of different values of χ will be μ, where

$$\mu = \frac{n(n+1)\ldots(n+r-1)}{1.2\ldots2},$$

and $\chi_1 f$, $\chi_2 f$.....$\chi_\mu f$ will form a covariantive plexus to f.

Again, let ϑ represent any product of the symbols

$$\frac{d}{dx_1}, \frac{d}{dx_2} \ldots \frac{d}{dx_n} \text{ of the degree } r.$$

$\vartheta_1 f$, $\vartheta_2 f$.....$\vartheta_\mu f$ will also form a covariant plexus to f.

The coefficients of connexion between the forms of either plexus depend in an analogous manner upon the coefficients of the substitution supposed to be impressed upon the variables, with the sole difference that every coefficient taken from the line r and column s of the determinant of substitution which appears in any coefficient of connexion of the one plexus is replaced by the coefficient taken from the line s and the column r in the corresponding coefficient of connexion for the other plexus.

Let $f(x, y)$ be any function of x, y of the degree $2m$

$$\left(\frac{d}{dx}\right)^m, \left(\frac{d}{dx}\right)^{m-1}\frac{d}{dy}, \text{ \&c.}\ldots\ldots \left(\frac{d}{dy}\right)^m$$

will form a covariantive plexus; thus, suppose

$$f(x, y) = a_1x^{2m} + 2mb_2x^{2m-1}y + \text{\&c.}\ldots\ldots + a_{2m+1}y^{2m}.$$

Then, omitting numerical factors, the plexus will be composed of the $(m+1)$ lines following:

$$\begin{aligned}
&a_1.x^m + ma_2.x^{m-1}y + \ldots\ldots + a_{m+1}.y^m,\\
&a_2.x^m + ma_3x^{m-1}.y + \ldots\ldots + a_{m+2}.y^m,\\
&\text{\&c.}\\
&a_{m+1}.x^m + ma_{m+2}.x^{m-1} + \ldots + a_{2m+1}.y^m,
\end{aligned}$$

and consequently, by the law of synthesis, the determinant

$$\left.\begin{array}{l}
a_1, a_2 \ldots\ldots a_{m+1}\\
a_2, a_3 \ldots\ldots a_{m+2}\\
\ldots\ldots\ldots\ldots\\
a_{m+1}, a_{m+2}.\ldots a_{2m+1}
\end{array}\right\} \text{ is an invariant of } f.$$

When this determinant is zero, I have proved in my paper on Canonical Forms, in the *Philosophical Magazine* for November last, that f is resoluble into the sum of $(m+1)$ powers

of linear functions of x and y. I shall hereafter refer to a determinant formed in this manner from the coefficients of f as its catalecticant. Mr. Cayley was, I believe, the first to observe that all catalecticants* are invariants.

Again, more generally, let $f(x, y, \xi, \eta)$ be a function of the m^{th} degree of x, y, and of a like degree in respect of ξ, η, which are supposed to be congredient with x and y,

$$f(x, y, \xi, \eta) + \lambda(x\eta - y\xi)^m$$

(say F) will be a concomitant of f; and therefore if we take the system

$$\left(\frac{d}{dx}\right)^m F, \left(\frac{d}{dx}\right)^{m-1} F, \ldots . \left(\frac{d}{dy}\right)^m F,$$

which will be functions of ξ and η alone, and take their resultant, this resultant will be an invariant of f. As a particular case of this theorem, let

$$f = \left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^m \phi,$$

whence ϕ is supposed to be a function of x and y only, and $2m$ dimensions f is a concomitant of ϕ, and therefore the invariant of f, obtained in the manner just explained, will be an invariant of ϕ. Thus then we have an instantaneous demonstration of the theorem given by me in the paper of the *Philosophical Magazine* before named, viz. if

$$\phi(x, y) = a_1 x^{2m} + 2ma_2 x^{2m-1}.y + \ldots\ldots + a_{2m+1}.y^{2m},$$

say, in order to fix the ideas, $= ax^6 + bx^5y + cx^4y^2 + \ldots gy^6$; then the determinant

$$\begin{vmatrix} a & b & c & d+\lambda \\ b & c & d-\frac{1}{3}\lambda & e \\ c & d+\frac{1}{3}\lambda & e & f \\ d-\lambda & e & f & g, \end{vmatrix}$$

and (the analogously formed determinant for the general case) will be an invariant of ϕ. The general determinant so formed is peculiarly interesting, because it furnishes when equated to zero the one sole equation necessary to be solved in order

* But the catalecticant of the biquadratic function of x, y was first brought into notice as an invariant by Mr. Boole; and the discriminant of the quadratic function of x, y is identical with its catalecticant, as also with its hessian. Meicatalecticizant would more completely express the meaning of that which, for the sake of brevity, I denominate the catalecticant.

to be able to effect the reduction of $\phi(x, y)$ to its canonical form, and gives the means, irrespective of any other view of the theory of invariants, of determining completely and absolutely the condition of the possibility of two given functions of the same degree of x, y being linearly transformable one into the other. This theorem will be obtained in a more general manner in the following section. I only pause now to make the very important observation, that not only is the determinant an invariant, but every minor system* of determinants that can be formed from it (there are of course m such systems) is an invariantive plexus to the given function (ϕ).

The form under which this theorem presents itself suggests a theorem vastly more general and of peculiar interest, as shewing a connexion between the theory of functions of a certain degree and of a certain number of variables with other functions of a lower degree but of a greater number of variables. Here again, under a different aspect, is reproduced the great principle of dialysis, which, originally discovered in the theory of elimination, in one shape or another pervades the whole theory of concomitance and invariants.

Let ϕ represent any function of the degree pq (of any number, or, to fix the idea, say of three variables x, y, z); let the general term of ϕ be represented by

$$\frac{pq(pq-1)\ldots 1}{(1.2\ldots\alpha)(1.2\ldots\beta)(1.2\ldots\gamma)}(a, \beta, \gamma)\, x^{\alpha}.y^{\beta}.z^{\gamma},$$

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* These minor systems mean as follows:—the system of r^{th} minors comprises all the distinct determinants that can be got by striking out from the square array (which I call the Matrix) from which the complete determinant is formed, any z lines and any z columns selected at will. The last, or m^{th} minor, is of course a system consisting of the coefficients of $\phi(x, y)$, and it is evident that if $\phi(x, y\ldots z)$ be any function of any number of variables $x, y\ldots z$, that the coefficients will form an invariantive plexus to ϕ.

The following remark as to the changes undergone by the coefficients of ϕ when the variables undergo any substitution, is not without interest and importance for the theory.

Let x become $fx + f'y + \ldots + (f)z$,

$y \ldots\ldots gx + g'y + \ldots + (g)z$,

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$z \ldots\ldots hx + h'y + \ldots + (h)z$.

Then the coefficient of the highest power of x becomes

$$\phi(f, g\ldots h),$$

and the coefficient of the term containing $y^r \ldots z^s$ becomes

$$\left(f'\frac{d}{df} + g'\frac{d}{dg} + \ldots + h'\frac{d}{dh}\right)^r \times \&c. \times \left\{(f)\frac{d}{d(f)} + (g)\frac{d}{d(g)} + \ldots + (h)\frac{d}{d(h)}\right\}^s .\phi(f, g\ldots h).$$

where $\alpha + \beta + \gamma = pq$, and (α, β, γ) represents a portion of the coefficient of $x^{\alpha}.y^{\beta}.z^{\gamma}$.

Let $$\frac{1.2\ldots p}{(1.2\ldots r)(1.2\ldots s)(1.2\ldots t)}\, x^r.y^s.z^t = \theta_{r,s,t},$$

where $r + s + t = p$, so that there are as many θ's as there are modes of subdividing p into three integral parts (zeros being admissible); i.e. $\frac{(p+1)(p+2)(p+3)}{1.2.3}$. Then any product such as $x^{\alpha}.y^{\beta}.z^{\gamma}$ may be divided in a variety of ways into the product of q of these θ's, and it may be shewn that the entire quantity

$$\frac{pq.(pq-1)\ldots 1}{(1.2\ldots\alpha)(1.2\ldots\beta)(1.2\ldots\gamma)}\,(x^{\alpha}.y^{\beta}.z^{\gamma})$$

$$= \Sigma\left\{\frac{1.2\ldots q}{(1.2\ldots m_1)(1.2\ldots m_2)\ldots(1.2\ldots m_r)}\,(\theta_{\mu_1}{}^{m_1}.\theta_{\mu_2}{}^{m_2}\ldots\theta_{\mu_s}{}^{m_s})\right\},$$

where $m_1 + m_2 + \ldots + m_s = q$. Consequently ϕ may be represented under the form of a function of the degree q of $\frac{(p+1)(p+2)(p+3)}{1.2.3}$ (say i) variables $\theta_1, \theta_2 \ldots\ldots \theta_i$, and its general term will be of the form

$$\frac{1.2\ldots q}{(1.2\ldots m_1)(1.2\ldots m_2)\ldots(1.2\ldots m_s)}\,(\alpha, \beta, \gamma)\,\{\theta^{m_1}.\theta^{m_2}\ldots\theta^{m_s}\},$$

where α, β, γ are the indices respectively of x, y, z, when the last factor is expressed as a function of these variables.* Now if ϑ be used to denote this new representation of ϕ when $\theta_1, \theta_2 \ldots \theta_2$ are treated as absolutely independent variables, and if we attach to it any universal concomitant, as $(x\xi + y\eta + z\zeta)^p$ admitting of being written under the form $\omega(\theta_1, \theta_2 \ldots \theta_i)$, wherein the coefficients will be functions of ξ, η, ζ; then any invariant to ϑ and ω, treated as two systems of ι variables, will be a concomitant to ϕ, the original function in x, y, z.† ϑ and ω may be termed respectively, for facility

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* See Note (1) in Appendix.

† In fact ϑ is a concomitant to ϕ and ω to a power of the universal concomitant; the θ's forming a system of variables congredient with the compound system $x^{\gamma_1}.y^{\gamma_1}.z^{\gamma_1}$, $x^{\gamma_2}.y^{\gamma_2}.z^{\gamma_2}$, &c.: and it must be well observed that the same substitutions which render ϑ and ω respectively identical with ϕ, and a power of the universal concomitant, would render an infinite number of other functions also coincident with the same; but none of these other functions would be concomitants. Herein we see the importance of the definition and conception of compound relation; the θ system being compound by relation with the x, y, z system, after the manner of cogredience.

of reference, the Particular and Absolute functions. Thus, for example, we take ϕ a function of x, y of the degree $4n$ (say $a_1.x^{4n} + 4n.a_2.n^{4n-1} + \&c. + a_{4n+1}$), and make $p = 2n$, $q = 2$, so that ϑ becomes a quadratic function of $(2n + 1)$ variables obtained by making $x^{2n} = \theta$, $x^{2n-1}y = \theta_2 \ldots y^{2n} = \theta_{2n+1}$,* and the concomitant ω, formed from $(\xi x + \eta y)^{2n}$, becomes

$$\theta_1\xi^{2n} + 2n.\theta_2.\xi^{2n-1}.\eta + \ldots . + \theta_{2n+1}.\eta^{2n};$$

and if we take R the quadratic invariant of ω, that is

$$R = \theta_1.\theta_{2n+1} - 2n\theta_2.\theta_{2n} \&c. \pm \frac{1.2.3\ldots(2n)}{(1.2\ldots n)^2}\tfrac{1}{2}(\theta_{n+1})^2,$$

it will readily be seen the determinant of $\vartheta + \lambda R$, treated as a quadratic function of $(2n + 1)$ variables, will give an invariant of ϕ, and this will be the same as that obtained by the particular method above given. Thus, suppose

$$\phi(xy) = ax^4 + 4bx^3y + bcx^2y^2 + 4dxy^3 + ey^4.$$

Let $$x^2 = \theta_1\ 2xy = \theta_1\theta_2\ y^2 = \theta_3,$$

$$\vartheta = a\theta_1^2 + 2b\theta_1\theta_2 + c\theta_2 + 2c\theta_1\theta_3 + 2d\theta_2\theta_3 + e\theta_2^2,$$

$$\omega = (x\xi + y\eta)^2 = x^2\theta_1 + xy\theta_2 + y^2\theta_3,$$

$$R = \theta_1\theta_3 - \frac{\theta_2^2}{4},$$

Then Λ the discriminant of $\vartheta + 2\lambda R$ in respect to $\theta_1\theta_2\theta_3$

$$= \begin{matrix} a & b & c + \lambda \\ b & c - \frac{1}{2}\lambda & d \\ c + \lambda & d & e \end{matrix}$$

and I may remark that the relations between the several transformees of the invariantive plexuses formed by the minor determinant systems of λ (in this, and in general for the case of an evenly-even index) may be found by treating $\vartheta + 2\lambda R$ as a quadratic function of the variables (in this case $\theta_1, \theta_2, \theta_3$) and applying the rule given by me in the *Philosophical Mag.* in my paper "On the relation between the Minor Determinants of linearly-equivalent Quadratic

* A slight variation upon the method as above explained for the general case has been here introduced inadvertently by writing $x^{2n-1}y = \theta_2$, &c., in lieu of $2nx^{2n-1}y = \theta_2$, &c., which, as it does not in any degree affect the reasoning, I have not deemed it worth while to alter.

Forms."* This second method, however, is not immediately applicable to the case of indices oddly even, *i.e.* of the form $4n+2$, to which the first method applies equally as to the case $4n$; for if we make $2n+1=p$ and $q=2$, ω being of an odd degree, has no quadratic invariant; it has however a quadratic covariant, which will be of the second degree in respect to $\theta_1, \theta_2 \ldots \theta_{p+1}$ as well as in respect to ξ, η; and if we call this R and take the discriminant of $\vartheta + \lambda R$ in respect to the variables $\theta_1, \theta_2 \ldots \theta_{p+1}$, we shall obtain, as I am indebted to a remark of my valued friend M. Hermite for bringing under my notice a very beautiful and interesting function of λ, of which all the coefficients will be contravariants of ϕ. Thus, let

$$\phi = ax^6 + 6bx^5y + 15cx^4y^2 + 20dx^3y^3 + 15ex^2y^4 + bfxy^5 + gy^6,$$

make $\quad x^3 = \theta_1,\ 3x^2y = \theta_2,\ 3xy^2 = \theta_3,\ y^3 = \theta_4,$

so that $\vartheta = a\theta_1^2 + 2b\theta_1\theta_2 + c\theta_2^2 + 2c\theta_1\theta_3 + 2d\theta_2\theta_3 + 2d\theta_1\theta_4$
$+ g\theta_4^2 + 2f\theta_3\theta_4 + e\theta_3^2 + 2e\theta_4\theta_2,$

$$\omega = (x\xi + y\eta)^3 = \theta_1\xi^3 + \theta_2\xi^2\eta + \theta_3\xi\eta^2 + \theta_4\eta^3,$$

$$R = \begin{vmatrix} 3\theta_1\xi + \theta_2\eta, & \theta_2\xi + \theta_3\eta \\ \theta_2\xi + \theta_3\eta, & \theta_3\xi + 3\theta_4\eta \end{vmatrix}$$

$$-R = \xi^2\theta_2^2 + \eta^2\theta_3^2 + \xi\eta\theta_2\theta_3 - 9\xi\eta\theta_1\theta_4 - 3\xi^2\theta_1\theta_3 - 3\eta^2\theta_2\theta_4.$$

Consequently the discriminant in respect to $\theta_1, \theta_2, \theta_3, \theta_4$ of $\vartheta - 2\lambda\Sigma$ becomes

a	b	$c - 3\lambda\xi^2$	$d - 9\lambda\xi\eta$
b	$c + 2\lambda\xi^2$	$d + \lambda\xi\eta$	$e - 3\lambda\eta^2$
$c - 3\lambda\xi^2$	$d + \lambda\xi\eta$	$e + 2\lambda\eta^2$	f
$d - 9\lambda\xi\eta$	$e - 3\lambda\eta^2$	f	g

* Moreover, upon the supposition made in the text, the particular and absolute functions I and w may be treated in all respects as if they were functions characterizing quadratic loci, and any singularity in their relation will correspond to and denote a singularity in the given function ϕ to which θ refers. Thus, for instance, if ϕ be a function of x, y of the eighth degree, θ and w will be quadratic functions of five letters each. Quadratic loci have no other singularity of relation than what corresponds to different species of contact. The number of contacts between loci, characterized by 5 letters, is 24 (see my paper in the *Phil. Mag.*, "On the contacts of lines and surfaces of the second order.") Consequently this mode of representing θ and w will give rise to the discovery and specification of 24 different kinds of singularity in ϕ, and the analytical characteristics of each of them. But there of course may, and in fact will, exist other singularities in ϕ besides those which have their correspondencies in the relations of these quadratic concomitants.

If this determinant be expanded as a function of λ, all the coefficients of the various powers of λ will be contravariants to the given function ϕ. The term involving λ^4 is zero. Let ξ become $-y$ and η become x, then the remaining terms (abstraction made of the powers of λ) become covariants of ϕ. The first terms (the coefficient of λ^3) becomes ϕ itself; the last term is the catalecticant, and thus we see, in general, that for functions of x and y of an oddly-even degree, a whole series of covariants may be interpolated between the function and its catalecticant, the dimensions in respect of the coefficients of ϕ in arriving at each step increasing by 1 unit and the degree in respect of the variables diminishing by 2 units. This is consequently a much simpler and more available scale than one with which I have been long previously acquainted, and which applies alike to functions of any even degree.

Thus, let $\phi(x, y)$ be of $2k$ dimensions; form all the even emanants of ϕ, which will be all of the form $\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^{2\iota} \cdot \phi$, and take their respective catalecticants in respect to ξ and η. We shall in this way obtain a regular scale of covariants interpolated between the Hessian of ϕ (corresponding to $\iota = 1$) and the catalecticant of ϕ (corresponding to $\iota = k$). If ϕ be of the degree $2k + 1$, we shall have an analogous scale interpolated between the Hessian of ϕ and its canonizant; the latter term denoting the function which is the product of the $k + 1$ linear functions of x and y, the sum of whose $(2k+1)^{\text{th}}$ powers is identically equal to ϕ.*

By means of the Theory of the Plexus we may obtain various representations of the same invariant; thus, for example, if we take F a function of x, y of the fifth degree and from its Hessian H, i.e.

$$\frac{d^2F}{dx^2}; \quad \frac{d^2F}{dx\,dy}$$

$$\frac{d^2F}{dy\,dx}; \quad \frac{d^2F}{dy^2}:$$

this will be a function of the sixth degree in x, y, and of the two orders in the coefficients. If we combine the two plexuses

$$\frac{dF}{dx}, \frac{dF}{dy}; \quad \frac{d^2H}{dx^2}, \frac{d^2H}{dx\,dy}, \frac{d^2H}{dy^2},$$

* See Note (2) in Appendix.

we shall have five equations between which $x^4, x^3y, x^2y^2, xy^3, y^4$ may be eliminated dialytically; the resultant will be of the $2+3.2$, *i.e.* the eighth order in the coefficients, and of the form $\square F - I_4^2$, where $\square F$ and I_4 are respectively the determinant and quintic invariant of F, each affected with a proper numerical multiplier (the "$B-A^2$" of my supplemental essay on canonical forms) which, as Mr. Cayley has remarked, may also be represented by the resultant of $P; \frac{dQ}{dx}; \frac{dQ}{dy}$ where P and Q are respectively the quadratic and cubic invariants in respect to ξ and η of $\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^4 . F$.

It will be well at this point to recapitulate in brief a method of elimination applicable to certain systems of functions published by me many years since in the *Philosophical Magazine*, and to compare this method with that afforded by the theory of the plexus for finding an invariant for each of the very same systems, possessing all the external characters, formed in a precisely similar manner to, and not impossibly identical *with*, the resultant of every such system. I shall devote my first moments of leisure to the ascertainment of this last most important point, as to the identity or otherwise of the plexus-invariant with the resultant. Take the case of three functions of x, y, z, (say ϕ, ψ, ω) each of the same degree n; to fix the ideas, suppose $n = 3$: there are two purely algebraical processes (modifications of the same method and leading to identical results) by which the resultant of ϕ, ψ, ω may be found. I shall call these processes the first and second, respectively.

First process: Write

$$\phi = x^2P + yQ + zR,$$
$$\psi = x^2P' + yQ' + zR',$$
$$\omega = x^2P'' + yQ'' + zR'':$$

[these decompositions may be effected in an infinite variety of manner, so that P, Q, R shall be integer functions of x, y, z]; take the linear resultant of ϕ, ψ, ω, in respect to x^2, y, z, which call $H_{2,1,1}$; this will evidently be of $9-4$, that is, of 5 dimensions. Form analogously the functions $H_{1,2,1}$, $H_{1,1,2}$; $H_{2,1,1}$, $H_{1,2,1}$, $H_{1,1,2}$ constitute an auxiliary system of functions which vanish when ϕ, ψ, ω vanish together; combine this auxiliary system with the augmentative system

$$\begin{array}{cccccc} x^2\phi & y^2\phi & z^2\phi & xy\phi & yz\phi & zx\phi \\ x^2\omega & y^2\omega & z^2\omega & xy\omega & yz\omega & zx\omega \\ x^2\psi & y^2\psi & z^2\psi & xy\psi & yz\psi & zx\psi. \end{array}$$

We shall thus have in all $3 + 3 \times 6$, that is, 21 functions into which the 21 terms x^5, x^4y, x^4z, &c. enter linearly: the linear resultant of these 21 functions is the resultant of ϕ, ψ, ω, clear of all extraneousness.

Second process: Write

$$\phi = x^3P + yQ + zR,$$
$$\psi = x^3P' + yQ' + zR',$$
$$\omega = x^3P'' + yQ'' + zR'',$$

and, as before, take the linear resultant $H_{3,1,1}$, which will however be of $9 - 5$, that is, of only 4 dimensions.

Again, take

$$\phi = x^2L + y^2M + zN,$$
$$\psi = x^2L' + y^2M' + zN',$$
$$\omega = x^2L'' + y^2M'' + zN'',$$

and form the determinant $H_{2,2,1}$; we shall thus have the auxiliary system

$$H_{3,1,1},\ H_{1,3,1},\ H_{1,1,3},\ H_{2,2,1},\ H_{2,1,2},\ H_{1,2,2}.$$

Let this be combined with the augmentative system

$$x\omega, y\omega, z\omega;\quad x\phi, y\phi, z\phi;\quad x\psi, y\psi, z\psi.$$

Between these $6 + 9$, that is, 15 functions, the 15 terms x^4, x^3y, x^3z, &c. may be linearly eliminated, and the resultant thus obtained will be precisely the same as that got by the preceding process.

Here we have 6 auxiliaries and 6 augmentatives; the auxiliaries are of three dimensions in respect to the coefficients of ϕ, ψ, ω; the augmentatives of one dimension only; in the former process there were 3 auxiliaries and 18 augmentatives, $6 \times 3 + 9 = 27 = 3 \times 3 + 18$.

Now let this method be compared with the following:

First process: Take the 18 augmentatives $x^2\phi$, $x^2\omega$, $x^2\psi$, &c. as in the first process of the algebraical method above explained; but in place of the 3 auxiliaries therein given, take another system of 9 as follows:

Write the determinant

$$\left.\begin{matrix} \frac{d\phi}{dx}, & \frac{d\phi}{dy}, & \frac{d\phi}{dz} \\ \frac{d\psi}{dx}, & \frac{d\psi}{dy}, & \frac{d\psi}{dz} \\ \frac{d\omega}{dx}, & \frac{d\omega}{dy}, & \frac{d\omega}{dz} \end{matrix}\right\} = R;$$

$\frac{dR}{dx}, \frac{dR}{dy}, \frac{dR}{dz}$ form a concomitantive plexus; the 18 augmentatives form another; the linear resultant of these two plexuses will be an invariant of ϕ, ψ, ω, and of precisely the same dimensions as the resultant last found; if they are not identical it will be indeed a matter of exceeding wonder, and even more interesting than if they should be proved so to be.

Second process: Combine the augmentative plexus

$$x\omega, y\omega, z\omega\,; \quad x\phi, y\phi, z\phi\,; \quad x\psi, y\psi, z\psi,$$

with the differential plexus

$$\frac{d^2R}{dx^2}, \frac{d^2R}{dx\,dy}, \frac{d^2R}{dy^2}, \frac{d^2R}{dy\,dz}, \frac{d^2R}{dz^2}, \frac{d^2R}{dz\,dx},$$

we thus obtain a linear resultant in a manner precisely similar to that afforded by the second process of our algebraical method.

In general, if ϕ, ψ, ω be of the degrees n, n, n, as there are two algebraical varieties of the linear method for finding the resultant, so are there two varieties of the concomitantive method for finding the resembling invariant. In both methods the augmentatives are identical; the only difference being in the auxiliary system.

In the first process the augmentative system will be got by operating upon each of the functions ϕ, ψ, ω, with the multipliers $x^{n-1}, y^{n-1}, z^{n-1}$, and the other homogeneous products of x, y, z; the auxiliary system by operating upon R with the symbolical multipliers $\left(\frac{d}{dx}\right)^{n-2}, \left(\frac{d}{dy}\right)^{n-2}, \left(\frac{d}{dz}\right)^{n-2}$, and the other homogeneous products of $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ of the degree $n-2$.

In the second process the augmentative system is formed by the aid of the multipliers $x^{n-2}, y^{n-2}, z^{n-2}$, &c., and the auxiliary system by aid of $\left(\frac{d}{dx}\right)^{n-1}, \left(\frac{d}{dy}\right)^{n-1}, \left(\frac{d}{dz}\right)^{n-1}$, &c.

For the particular case of $x = 2$ the first process of the concomitantive method is merely an application under its most symmetrical form of the first process of the general algebraical method. The second process of the concomitantive method for this same case (at least when ϕ, ψ, ω are the partial differential coefficients of the same function of the third degree) has been shewn by Dr. Hesse to give the resultant, so that for this case, at all events, we know that each concomitantive auxiliary must be a linear function of the augmentatives and the algebraical auxiliaries.

Again, if we go to the system where ϕ, ψ, ω are of the respective degrees $n, n, n+1$. In the algebraical method (for applying which there are no longer two, but one only process), the augmentative system is obtained by multiplying ϕ by the homogeneous products of $x^{n-1}, x^{n-1}y, x^{n-1}z$, &c., ψ by the like procduts, and ω by the homogeneous products x^{n-2}, $x^{n-2}y$, &c.

The auxiliary system is made up of functions of the general form $H_{p,q,r}$ where $p + q + r = n + 2$,

$H_{p,q,r}$ being the determinant obtained by writing

$$\phi = Lx^p + My^q + Nz^r,$$
$$\psi = L'x^p + M'y^q + N'z^r,$$
$$\omega = L''x^p + M''y^q + N''z^r;$$

and in like manner for the case of ϕ, ψ, ω, being of the respective degrees $n, n, n-1$, the augmentative system is obtained by affecting ϕ, ψ each with multipliers $x^{n-2}, x^{n-2}y$, &c., and ω with the multipliers $x^{n-1}, x^{n-1}y$, &c.

The number of functions (for either case) in the augmentative and auxiliary plexuses thus obtained will be found to be exactly equal to the number of terms in each such function, as shewn by me in the paper alluded to. Let this be compared with the transcendental method (I use this word at this point in preference to concomitantive, because in fact the algebraical and differential auxiliary systems are both alike concomitantive plexuses to ϕ). For the case of $n, n, n+1$, R the Jacobian determinant of ϕ, ψ, ω will be of the degree $3n - 2$, and the system $\left(\frac{d}{dx}\right)^{n-1} R, \left(\frac{d}{dx}\right)^{n-2}\cdot\left(\frac{d}{dy}\right) R$, &c. combined with the augmentative systems

$$x^{n-2}\omega,\ x^{n-1}y\omega,\ \&c.$$
$$x^{n-1}\phi,\ x^{n-2}y\phi,\ \&c.$$
$$x^{n-1}\psi,\ x^{n-2}y\psi,\ \&c.$$

will give an invariant resembling (at least in generation and form) if not identical with the resultant of ϕ, ψ, ω. For the case of ϕ, ψ, ω being of the degrees $n, n, n-1$, the Jacobian R is of the degree $3n-4$ and $\left(\frac{d}{dx}\right)^{n-2} R, \left(\frac{d}{dx}\right)^{n-3} \frac{d}{dy} R$, &c. is the system which, combined with the augmentative systems

$$x^{n-2}\phi,\ x^{n-3}y\phi,\ \&c.$$
$$x^{n-2}\psi,\ x^{n-3}y\psi,\ \&c.$$
$$x^{n-1}\omega,\ x^{n-2}y\omega,\ \&c.$$

will produce the resembling invariant.

Finally, for the last and more special case which the algebraical method applies to, viz. of $\phi, \psi, \omega, \theta$, four quadratic functions of x, y, z, t, there can be here little doubt (upon the first impression) that in place of the algebraically obtained plexus

$$H_{2,1,1,1},\ H_{1,2,1,1},\ H_{1,1,2,1},\ H_{1,1,1,2},$$

may be substituted the differential plexus

$$\frac{dR}{dx},\ \frac{dR}{dy},\ \frac{dR}{dz},\ \frac{dR}{dt},$$

which, combined with the augmentatives

$$x\phi, x\psi, x\omega, x\theta;\ y\phi, y\psi, y\omega, y\theta;\ z\phi, z\psi, z\omega, z\theta;\ t\phi, t\psi, t\omega, t\theta,$$

will render possible the dialytic elimination of the 20 homogeneous products

$$x^3, x^2y, x^2z, x^2t, xyz, y^3, \&c.\ \&c.^*$$

Upon precisely the same principles may be verified instantaneously the method given by Hesse (without demonstration) for finding the polar reciprocal of lines of the third and fourth orders, at least to the extent of seeing that the functions obtained by his methods are contravariants (of the right degree and order) of the function from which they are derived. The polar reciprocal to a *surface* of the third degree may be obtained in the same manner.

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* Subsequent reflection induces me to reject as very improbable the (at first view likely) conjecture of the identity of the resultant with the invariant which simulates its form, except in the proved cases of three quadratic functions and the strongly resembling case of four quadratic functions last adverted to in the text above. Did this identity obtain, analogy would indicate that the catalecticant of the Hessian of two homogeneous functions of the same degree in x, y should be identical with their resultant, which is easily demonstrated to be false, except when the functions are of the third degree.

Let $\phi(x, y, z, t)$ be the characteristic of such a surface. If we form a differential plexus of the first emanant of ϕ taken together with the concomitant $W(= x\xi + y\eta + z\zeta + t\theta)$, or operating with

$$\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \frac{d}{dt} \text{ upon } \left(\xi' \frac{d}{dx} + \eta' \frac{d}{dy} + \zeta' \frac{d}{dz} + \theta' \frac{d}{dt}\right)(\phi + \lambda w),$$

and combining this plexus with $x\xi' + y\eta' + z\zeta' + t\theta'$, the resultant taken in respect to $\xi', \eta', \zeta', \theta'$ (say R) will (according to the law of synthesis) be a contravariant to the system $\phi + \lambda W$ and w, and therefore to ϕ, because w is itself a concomitant to ϕ. R is of the third degree in x, y, z, t, as also in the coefficients of ϕ. If we form a differential plexus of $R + \mu w$ analogous to that formed above with $\phi + \lambda w$, and combine these two plexuses with the augmentative system xw, yw, zw, tw, there will be $4 + 4 + 4$, that is, 12 functions containing the 12 terms $x^2, y^2, z^2, t^2, xy, xz, xt, yz, yt, zt, \lambda, \mu$, and the dialytic resultant, which will be found to be a contravariant of the twelfth degree in ξ, η, ζ, θ, and of the twelfth order in respect of the coefficients of ϕ, will be (there can be little doubt) the polar reciprocal to the characteristic ϕ. A few remarks upon the analytical character of a polar reciprocal may be not here out of place. If ϕ be any homogeneous function of the degree m of any number (n) of variables $(x, y \ldots z)$, the object of the theory of polar reciprocals is to discover what is the relation between $\xi, \eta \ldots \zeta$ expressed in the simplest terms such that when this equation is satisfied $\xi x + \eta y + \ldots + \zeta z = 0$ will be tangential to $\phi = 0$. In order for this to take effect it is necessary that when any one of the variables z is expressed in terms of the others $\ldots y, x$, and this value established in ϕ, then the discriminant of ϕ, so transformed, must be zero. Consequently the characteristic of the polar reciprocal to ϕ is that rational integral function which is common to all the discriminants obtained by expressing ϕ (by aid of the equation $\xi x + \eta y + \ldots + z\zeta$) as a function of any $(n-1)$ of the variables. Let I be any invariant whatever of the order r of ϕ (meaning by $\overset{x}{I}$ this last symbol what ϕ becomes when x is eliminated), and $\overset{y}{I} \ldots \overset{z}{I}$ the corresponding invariants when $y \ldots z$ respectively are eliminated; I will evidently be of the form $\frac{E}{(\xi)^{mr}}$, the numerator being an integer of r dimensions in the coefficients of ϕ and of mr dimensions in respect of $\xi, \eta \ldots \zeta$;

and by the fundamental definition of invariants it may easily be shewn that

$$I_x : I_y : \ldots : I_z :: \frac{1}{\xi^{n-1}} : \frac{1}{\eta^{n-1}} : \ldots : \frac{1}{\zeta^{n-1}},^*$$

and therefore

$$\frac{E_x}{\xi^p} = \frac{E_y}{\eta^p} = \ldots = \frac{E_z}{\zeta^p}, \quad \text{where } p = \frac{m(n-2)r}{n-1}.$$

Consequently all these quotients must be essentially integer, and any one of them will be of the order r in respect of the coefficients of ϕ and of the degree $\frac{mr}{n-1}$ in respect of $\xi, \eta \ldots \zeta$. Consequently the polar characteristic of ϕ, which is the common factor of the *discriminants* of $I_x, I_y \ldots I_z$ (for which species of invariant r evidently is equal to $(n-1)(m-1)^{n-2}$, being in fact the discriminant of a function of the m^{th} degree of $(n-1)$ variables) will be of the order $(n-1)(m-1)^{n-2}$ in respect of the coefficients of ϕ and of the degree $m(m-1)^{n-2}$ in respect of the contragredients $\xi, \eta \ldots \zeta$.

As to what relates to the reciprocity which exists between ϕ and its polar reciprocal ψ, this is included in a much higher theory of elimination, one proposition of which may be enunciated somewhat to the effect following, viz. that if ϕ be a homogeneous function of $x, y \ldots z$, and ω of $x, y \ldots z$, $u, v \ldots w$, and if, by aid of the equations

$$\phi = 0,$$

$$\frac{d\phi}{dx} + \lambda \frac{d\omega}{dx} = 0,$$

$$\frac{d\phi}{dy} + \lambda \frac{d\omega}{dy} = 0,$$

&c.

$$\frac{d\phi}{dz} + \lambda \frac{d\omega}{dz} = 0,$$

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* We see indirectly from this, that for a function of $(n-1)$, say γ variables of the degree m, an invariant of the order r must be subject to the condition that $\frac{mr}{\gamma}$ = an integer. This is easily shewn upon independent grounds; when $\gamma = 2$ $\frac{mr}{\gamma}$ must be not merely an integer but an *even integer*, and doubtless some analogous law applies to the general case.

$x, y \ldots z$ be eliminated and the resultant be called ψ, then the effect of performing a similar operation upon ψ, ω, with respect to $u, v \ldots w$, as that just above indicated for the system ϕ, ω, with respect to $x, y \ldots z$, will be to give a resultant, one factor of which will be the primitive function ϕ over again. There is some reason for supposing that polar reciprocals, which are scarcely ever (if ever, except indeed for quadratic functions) the simplest contravariants to a given function, may be expressed algebraically by means of the simpler contravariants, in the same way as discriminants admit in many, if not in all cases, with the same exception as above) of being represented as algebraical functions of invariants of a lower order or simpler form.

I close this section with the remark that every complete and unambiguous system of functions of the constants in a given form or set of forms *characteristic** of any singularity absolute or relative in such form or forms must constitute an invariantive plexus or set of invariantive plexuses. The system unambiguously characteristic of a singularity of an order n will (except when $n = 1$) almost universally consist of far more than n functions, subject of course to the existence of syzygetic† relations between any $(n + 1)$ of such functions. The existence of multiple roots of a function of two variables is a specific, but by no means a peculiar case of singularity, and requires, for its complete and systematic elucidation, to be treated in connexion with the general theory of the subject.

Sect. III.—*On Commutants.*

The simplest species of commutant is the well-known common determinant.

If we combine each of the n letters $a, b \ldots l$ with each of the other n, $\alpha, \beta \ldots \lambda$, we obtain n^2 combinations which may be used to denote the terms of a determinant of n lines and

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* I repeat here that a function or system of functions which severally equated to zero express unequivocally and completely the existence of any position or negation, is termed the characteristic of such position or negation. Thus (*ex. gr.*) the resultant of a group of equations is the characteristic of the possibility of their coexistence. The discriminant of a function of two variables is the characteristic of its possession of two equal factors; the catalecticant is the characteristic of its decomposability into the sum of a defined number of powers of linear functions of the variables, &c.

† Rational integer functions which admit of being multiplied severally by other rational integer functions such that the sum of the products is identically zero, are said to be 'syzygetically related.'

columns, as thus:

$$a\alpha,\ a\beta \ldots a\lambda,$$
$$b\alpha,\ b\beta \ldots b\lambda,$$
$$\ldots\ldots\ldots\ldots$$
$$l\alpha,\ l\beta \ldots l\lambda.$$

It must be well understood that the single letter of either set are mere umbræ, or shadows of quantities, and only acquire a real signification when one letter of one set is combined with one of the other set. Instead of the inconvenient form above written, we may denote the determinant more simply by the matrix

$$a,\ b,\ c \ldots l,$$
$$\alpha,\ \beta,\ \gamma \ldots \lambda;$$

and to find the expanded value of such a matrix the rule is evidently to take one of the lines in all its $1.2.3 \ldots n$ different forms, arising from the permutations of the letters (or umbræ) which it contains; and then form the product of the n quantities formed by the combination of the respective pairs of letters in the same vertical column, affecting such product with the sign of + or − according to the rule, that all products corresponding to arrangements of the terms subject to the permutation derivable from one another by an even number of interchanges are of the same, and by an odd number of interchanges of a contrary sign. If both lines are permuted and a similar rule applied, with the additional circumstance that the sign of the products is made to depend on the product of the algebraical signs due to the respective arrangements in the two lines of umbræ, it is evident that the result will be the same as when only one line is put into motion, save and except that a numerical factor $1.2.3 \ldots n$ will affect each term. If the two sets of umbræ $a, b, c \ldots l$; $\alpha, \beta, \gamma \ldots \lambda$ be taken identical, and if it be convened that the order of the combination of any two letters shall not affect the value of the quantity thereby denoted, $\begin{matrix} a, b, c \ldots l \\ a, b, c \ldots l \end{matrix}$ will denote a symmetrical determinant.

If instead of two lines of umbræ, three or more be taken, the same principle of solution will continue to be applicable. Thus, if there be a matrix of any even number r of lines each of n umbræ,

$$a_1,\ b_1 \ldots l_1,$$
$$a_2,\ b_2 \ldots l_2,$$
$$\ldots\ldots\ldots$$
$$a_s,\ b_s \ldots l_s,$$

the first may be supposed to remain stationary, and the remaining $(r-1)$ lines each be taken in $1.2...n$ different orders; every order in each line will be accompanied by its appropriate sign + or −; and each different grouping in each line will give rise to a particular grouping of the letters read off in columns. The value of the commutant expressed by the above matrix will therefore consist of the sum of $(1.2...n)^{r-1}$ terms, each term being the product of n quantities respectively symbolized by a group of r letters and affected with the sign + or − according as the number of negative signs in the total of the arrangements of the lines (from the columnar reading off of which each such term is derived) is even or odd.

For example, the value of

$$\begin{matrix} a & b \\ c & d \\ e & f \\ g & h \end{matrix}$$

will be found by taking the $(1.2)^3$ arrangements, as below,

$$\begin{matrix} a\,b & a\,b & a\,b & a\,b & a\,b & a\,b & a\,b & a\,b \\ c\,d & d\,c & c\,d & d\,c & c\,d & d\,c & c\,d & d\,c \\ e\,f & e\,f & f\,e & f\,e & e\,f & e\,f & f\,e & f\,e \\ g\,h & g\,h & g\,h & g\,h & h\,g & h\,g & h\,g & h\,g \end{matrix}$$

The signs of c, d; e, f; g, h being supposed +, those of d, c; f, e; and h, g will be each −. Consequently the sum of the terms will be expressed by

$$aceg \times bdfh - adeg \times bcfh - acfg \times bdef + adfg \times bceh,$$
$$- aceh \times bdfg + adeh \times bcfg + acfh \times bdeg - adfh \times bceg.$$

Commutants thus formed may be termed total commutants, because the entire of each line is made to pass through all its possible forms of arrangement. In total commutants it is necessary that the number of lines r be even; for if taken odd, on making all the r lines to change instead of obtaining $1.2...n$ lines, the result obtained when all but one are made to change, it will be found that the latter will be repeated $\frac{1}{2}(1.2...n)$ times with the sign +, and $\frac{1}{2}(1.2...n)$ times with the sign −, so that the algebraical sum of the terms will be zero. Moreover the commutants of the species above described, besides being total, are Simple, inasmuch as all the umbræ to be termed consist of single letters.

My first proposition in the application of the theory of commutants to that of forms is as follows:

Let ϕ be a function homogeneous and linear in respect to an even number r of any systems whatever of variables, as

$$x_1, y_1 \ldots t_1 ; \quad x_2, y_2 \ldots t_2 ; \quad x_r, y_r \ldots t_r .$$

Form the commutant

$$\begin{array}{cccc} \frac{d}{dx_1}, & \frac{d}{dy_1} & \ldots\ldots & \frac{d}{dt_1}, \\ \frac{d}{dx_2}, & \frac{d}{dy_2} & \ldots\ldots & \frac{d}{dt_2}, \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \frac{d}{dx_r}, & \frac{d}{dy_r} & \ldots\ldots & \frac{d}{dt_r}. \end{array}$$

Let the general term of this commutant, expanded, be called

$$F_{\theta_1} \times F_{\theta_2} \times \ldots \times F_{\theta_r} \text{ then is } \Sigma + F_{\theta_1}.\phi \times F_{\theta_2}.\phi \times \ldots \times F_{\theta_r}.\phi$$

a covariant or invariant, as the case may be, of ϕ.

Be it observed that the march of the substitution for the different sets of variables in the above proposition is supposed to be perfectly independent. All the systems but one may undergo linear transformation, or they may all undergo distinct and disconnected transformations at the same time, and the proposition still continue applicable. It will however evidently be no less applicable should the march of substitution for any of the systems become congredient or contragredient to that of any other systems.

If we suppose ϕ to be a function of an even degree r of a single system of n variables $x, y \ldots t$, so that the r systems x_1, y_1, &c. x_2, y_2, &c. $\ldots x_r, y_r$, &c. become identical, we can at once infer from the above scheme the existence and mode of forming an invariant to ϕ of the order n. This last appears for the case $n = 2$, and ought, for all other values of n, to have been known* to the author of the immortal discovery

* That this was not known explicitly to and should have escaped the penetration of the sagacious author of the theory, and those who had studied his papers, must be attributed to the imperfection of the notation heretofore employed for denoting the coefficients of a homogeneous polynomial function. The umbral method of denoting such a function ϕ of the degree r under the form of $(ax + by + \ldots + cz)^r$, which is equivalent to, but a more compendious and independent mode of mentally conceiving and handling the representation

$$\left(x \frac{d}{dx} + y \frac{d}{dy} + \ldots\ldots + z \frac{d}{dz} \right) \phi$$

exhibits the true internal constitution of such functions, and necessarily leads to the discovery of their essential properties and attributes.

of invariants, termed by him hyperdeterminants, in the sense which, according to the nomenclature here adopted, would be conveyed by the term hyperdiscriminants.

Before proceeding to discuss the theory of compound total computants, or enlarging upon that of partial commutants, I shall make an interesting application of the preceding general proposition to the discovery of Aronhold's S and T, the two invariants respectively of the fourth and sixth order appertaining to a homogeneous cubic function (say F) of three variables x, y, z. These may be termed respectively H_4 and H_6. As to H_6 a theoretically possible but eminently prolix and ungraceful method immediately presents itself, viz. to take $F^2 = G$, and after forming the commutant with six lines,

$$\begin{matrix} \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \end{matrix}$$

to operate with the 6^5 ternary products of which this is made up upon G: the result being an invariant of G, will be so to F, and being of the third degree in respect to the coefficients of G, will be of the sixth in respect to those of F. It will evidently therefore be H_6, or at least a numerical multiple of H_6, the form of which, inasmuch as the only other invariant is H_4, we know in form to be unique. But the general theorem affords another and probably the most practically compendious* solution as regards H_6, of which the question admits.

* Having since this was printed been favored with a view of some of the proof-sheets of Mr. Salmon's most valuable Second Part of his System of Analytical Geometry (about to appear, and which is calculated, in my opinion, to awaken a higher idea of and excite a new taste for geometrical researches in this country), I find that I am mistaken in this point; the less symmetrical method operated with by Mr. Salmon being decidedly the shortest for practically obtaining S and T in the general case. Symmetry, like the grace of an eastern robe, has not unfrequently to be purchased at the expense of some sacrifice of freedom and rapidity of action.

Let G^* represent the mixed concomitant to F formed by the bordered determinant

$$\begin{array}{cccc} \frac{d^2F}{dx^2} & \frac{d^2F}{dx\,dy} & \frac{d^2F}{dx\,dz} & \xi \\ \frac{d^2F}{dy\,dx} & \frac{d^2F}{dy^2} & \frac{d^2F}{dy\,dz} & \eta \\ \frac{d^2F}{dz\,dx} & \frac{d^2F}{dz\,dy} & \frac{d^2F}{dz^2} & \theta \\ \xi & \eta & \theta & 0 \end{array}$$

G is a function of the second order as to x, y, z, and of the like order in respect to ξ, η, θ, which two systems will be respectively congredient and contragredient in respect to the x, y, z system in F. In other words, which is all we need to look to, G is a concomitant of F, and so also will be

$$G + \lambda(x\xi + y\eta + z\zeta)^2,$$

which may be termed H. Form now the commutant

$$\begin{array}{ccc} \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ \frac{d}{d\xi} & \frac{d}{d\eta} & \frac{d}{d\zeta} \\ \frac{d}{d\xi} & \frac{d}{d\eta} & \frac{d}{d\theta} \end{array}$$

this being applied to H will give an invariant (the fact that the march of the substitutions for the systems x, y, z; ξ, η, ζ is contrary, being completely immaterial to the applicability of the general theorem above given); the commutant so formed will be a cubic function of λ, in which the coefficient of λ^3 is a numerical quantity, that of λ^2 is zero, that of λ is H_4 and the constant term is H_6.

Thus (*ex. gr.*) let $F = x^3 + y^3 + z^3 + 6mxyz$, then

$$G = \begin{vmatrix} x & mz & my & \xi \\ mz & y & mx & \eta \\ my & mx & z & \zeta \\ \xi & \eta & \zeta & \theta \end{vmatrix}$$

* G is the mixed concomitant to the given cubic function, which is halfway (so to speak) between it and its polar reciprocal. In fact, when the operation is repeated upon G, which was executed upon the given function to obtain G (that is, when we *border* the Hessian of G in respect to x, y, z, vertically and horizontally with the column and line ξ, η, ζ) the determinant thereby represented becomes the polar reciprocal to the given function.

and therefore

$$H = \Sigma\{(\lambda - m^2)\, x^2\xi^2 + (\lambda + m^2)\, 2yz\eta\zeta + yz\xi^2 - 2mx^2\eta\zeta\},$$

the Σ implying the sum of similar terms with reference to the interchanges between x, ξ; y, η; z, ζ.

In developing the commutant above, the first line may be kept in a fixed position; for the sake of brevity, (x), (y), (z); (ξ), (η), (ζ) may be written in the place of

$$\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}; \quad \frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta},$$

and it will readily be seen that the only effective arrangements will be as underwritten:

$$\left.\begin{array}{l|lll} (x)(y)(z) & (x)(y)(z) & (x)(y)(z) \\ (x)(y)(z) & (x)(y)(z) & (x)(y)(z) \\ (\xi)(\eta)(\zeta) & (\eta)(\zeta)(\xi) & (\zeta)(\xi)(\eta) \\ (\xi)(\eta)(\zeta) & (\zeta)(\xi)(\eta) & (\eta)(\zeta)(\xi) \end{array}\right\}$$

$$\begin{array}{llllll} (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) \\ (x)(z)(y) & (x)(z)(y) & (\zeta)(\eta)(\xi) & (z)(y)(x) & (y)(x)(z) & (y)(x)(z) \\ (\xi)(\eta)(\zeta) & (\xi)(\zeta)(\eta) & (\xi)(\eta)(\zeta) & (\zeta)(\eta)(\xi) & (\xi)(\eta)(\zeta) & (\eta)(\xi)(\zeta) \\ (\xi)(\zeta)(\eta) & (\xi)(\eta)(\zeta) & (\zeta)(\eta)(\xi) & (\xi)(\eta)(\zeta) & (\eta)(\xi)(\zeta) & (\xi)(\eta)(\zeta) \\ (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) \\ (x)(z)(y) & (x)(z)(y) & (z)(x)(y) & (z)(y)(x) & (y)(x)(z) & (y)(x)(z) \\ (\eta)(\zeta)(\xi) & (\zeta)(\eta)(\xi) & (\xi)(\zeta)(\eta) & (\zeta)(\xi)(\eta) & (\eta)(\zeta)(\xi) & (\xi)\{\zeta)(\eta) \\ (\zeta)(\eta)(\xi) & (\eta)(\zeta)(\xi) & (\zeta)(\xi)(\eta) & (\xi)(\zeta)(\eta) & (\xi)(\zeta)(\eta) & (\eta)(\zeta)(\xi) \end{array}$$

$$\left.\begin{array}{l|lll} (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) \\ (y)(z)(x) & (z)(x)(y) & (y)(z)(x) & (y)(z)(x) & (z)(x)(y) & (z)(x)(y) \\ (\zeta)(\xi)(\eta) & (\eta)(\zeta)(\xi) & (\xi)(\eta)(\zeta) & (\eta)(\zeta)(\xi) & (\xi)(\eta)(\zeta) & (\zeta)(\xi)(\eta) \\ (\xi)(\zeta)(\eta) & (\eta)(\zeta)(\xi) & (\eta)(\zeta)(\xi) & (\xi)(\eta)(\zeta) & (\zeta)(\xi)(\eta) & (\xi)(\eta)(\zeta) \end{array}\right\}$$

The signs of the four lines in each of these arrangements are two alike, and two contrary to the signs of the correspondent lines in the first arrangement; hence the effective sign is the same for all, and the result, after rejecting from each term the common factor -16, is seen, from inspection, to be

$$4(\lambda - m^2)^3 - 8m^3 + 6(\lambda - m^2)(\lambda + m^2)^2 - 12m(\lambda + m^2) + 2(\lambda + m^2)^3 + 1,$$

which is equal to

$$12\lambda^3 + 0.\lambda^2 - 12(m - m^4)\lambda + 1 - 20m^3 - 8m^6,$$

$m - m^4$ and $1 - 20m^3 - 8m^6$ are the two invariants (Aronhold's S and T) for the canonical form operated upon; and it will be observed that

$$(1 - 20m^3 - 8m^6)^2 + 64(m - m^4)^3 = (1 + 8m^3)^3,$$

which is easily proved to be the discriminant of

$$x^3 + y^3 + z^3 + 6mxyz.$$

It may however be observed, that this is not the discriminant of the function in λ just found, as reasons of analogy* might have suggested it probably would be: in order that this might be the case, the coefficient of λ^3 should be 4 instead of 12, and of λ, $m - m^4$ instead of $m^4 - m$. There is ground for supposing that another function of λ may be found by a different method, in which this relation will take effect.

The theorem above given for simple total commutants admits of an interesting application to the general case of a function F of the n^{th} degree, in respect to each of two independent systems of two variables x, y; ξ, η. Let F be symbolically represented by $(ax+by)^n.(a\xi+\beta\eta)^n$, so that $a^n.a^n$ represents the coefficient of $x^n\xi^n$, $na^{n-1}.ba^n$ of $x^{n-1}.y\xi^2$, &c. &c.; then the commutant

$$\begin{array}{l} a\,b \ \ldots\ (1), \\ a\,b \ \ldots\ (2), \\ \cdots \\ a\,b \ \ldots\ (n), \\ a\,\beta \ \ldots\ (1), \\ a\,\beta \ \ldots\ (2), \\ \cdots \\ a\,\beta \ \ldots\ (n), \end{array}$$

will represent a quadratic invariant of F, which will contain $(n+1)^2$ coefficients. By expanding this commutant we obtain a general expression for the invariant under a very interesting form.

I now proceed to give the general theorem for compound total commutants as applicable to the discovery of invariants.

Let there be a function of m disconnected classes of systems of variables: let the systems in the same class be supposed all distinct but congredient with one another. The function is supposed to be linear in respect to each system in each class, and the number of systems is the same for all the classes, and the number of variables the same in each

* The biquadratic function of x, y having only one parameter, and therefore two invariants, its theory possesses striking analogies to the theory of the cubic function of three letters. The function in λ which gives these invariants for the first-named function, according to the method given in the first section, has the same discriminant as the function itself.

system. This function may then be represented symbolically under the form

$$({}^1a_1.{}^1x_1 + {}^1b_1.{}^1y_1 + \ldots + {}^1l_1.{}^1t_1)({}^1a_2.{}^1x_2 + {}^1b_2.{}^1y_2 + \ldots + {}^1l_2.{}^1t_2)$$
$$\ldots\ ({}^1a_n.{}^1x_n + {}^1b_n.{}^1y_n + \ldots\ {}^1l_n.{}^1t_n)$$
$$\times ({}^2a_1.{}^2x_1 + {}^2b_1.{}^2y_1 + \ldots + {}^2l_1.{}^2t_1)({}^2a_2.{}^1x_2 + {}^2b_2.{}^1y_2 + \ldots + {}^2l_2.{}^1t_2)$$
$$\ldots\ ({}^2a_n.{}^1x_n + {}^2b_n.{}^1y_n + \ldots\ {}^2l_n.{}^1t_n)$$
$$\times \text{ \&c.}$$
$$\times ({}^pa_1.{}^px_1 + {}^pb_1.{}^py_1 + \ldots + {}^pl_1.{}^pt_1)({}^pa_2.{}^px_2 + {}^pb_2.{}^py_2 + \ldots + {}^pl_2.{}^pt_2)$$
$$\ldots\ ({}^pa_n.{}^px_n + {}^pb_n.{}^py_n + \ldots\ {}^pl_n.{}^pt_n).$$

In this expression the $x, y, \ldots\ t$'s are all real, but the $a, b, \ldots\ l$'s all umbral; in fact, ${}^1a_g, {}^1b_g$, &c. may be understood to denote $\frac{d}{d^1x_g}$, $\frac{d}{d^1y_g}$, &c.

The n systems in each of variables in each of the lines above written are supposed to be congredient *inter se*.

Take the symbolical product of the first line, first making *for the moment*

$${}^1x_1 = {}^1x_2 = \ldots\ {}^1x_n = x, \text{ \&c. \&c., } {}^1t_1 = {}^1t_2 = \ldots = {}^1t_n = t;$$

and let the coefficients of the several terms

$$x^n,\ x^{n-1}.y\ \ldots\ \text{\&c.},$$

be called $${}^1A_1\ \ {}^1A_2\ \ \ldots\ {}^1A_\mu,$$

where μ is the number of terms contained in a homogeneous function of the n^{th} degree of the m variables $x, y\ \ldots\ t$. In like manner proceed with each of the lines, and then write down the commutant

$$\begin{matrix} {}^1A_1 & {}^1A_2 & \ldots & {}^1A_\mu \\ {}^2A_1 & {}^2A_2 & \ldots & {}^2A_\mu \\ \ldots & \ldots & \ldots & \ldots \\ {}^pA_1 & {}^pA_2 & \ldots & {}^pA_\mu. \end{matrix}$$

This commutant is an invariant of F: it will of course be remembered that, unless p is even, the commutant vanishes.

Thus, for example, take two sets of two systems of two variables: in all four systems,

$$x, y;\ \ \xi, \eta\ :\ p, q;\ \ \phi, \psi,$$

each couple of systems on either side of the colon (:) being congredient *inter se:* and let F be symbolically represented by

$$(ax + by)(a\xi + \beta\eta)(lp + mq)(\lambda\phi + \mu\psi);$$

then the invariant given by the theorem will be the commutant

$$aa\,;\quad a\beta + ab\,;\quad b\beta,$$
$$l\lambda\,;\quad l\mu + \lambda m\,;\quad m\mu.$$

The six positions of which are as below written (the first three being positive and the second three negative)

$aa\,;\ a\beta+ab\,;\ b\beta$ $\quad$ $aa\,;\ a\beta+ab\,;\ b\beta$ $\quad$ $aa\,;\ a\beta+ab\,;\ b\beta$
$l\lambda\,;\ l\mu+\lambda m\,;\ m\mu$ $\quad$ $l\mu+\lambda m\,;\ m\mu\,;\ l\lambda$ $\quad$ $m\mu\,;\ l\lambda\,;\ l\mu+\lambda m$
$aa\,;\ a\beta+ab\,;\ b\beta$ $\quad$ $aa\,;\ a\beta+ab\,;\ b\beta$ $\quad$ $aa\,;\ a\beta+ab\,;\ b\beta$
$l\mu+\lambda m\,;\ l\lambda\,;\ m\mu$ $\quad$ $l\lambda\,;\ m\mu\,;\ l\mu+\lambda m$ $\quad$ $m\mu\,;\ l\mu+\lambda m\,;\ l\lambda.$

If we write F under its explicit form,

$$\begin{aligned} &Ax\xi p\phi + Bx\xi p\psi + Cx\xi q\phi + Dx\xi q\psi \\ &+ A'x\eta p\phi + B'x\eta p\psi + C'x\eta q\phi + D'x\eta q\psi \\ &+ A''y\xi p\phi + B''y\xi p\psi + C''y\xi q\phi + D''y\xi q\psi \\ &+ A'''\eta\xi p\phi + B'''y\eta p\psi + C'''y\eta q\phi + D'''y\eta q\psi, \end{aligned}$$

we have identically the relations following,

$aal\lambda = A$ $\quad$ $aal\mu = B$ $\quad$ $aam\lambda = C$ $\quad$ $aam\mu = D$
$a\beta l\lambda = A'$ $\quad$ $a\beta l\mu = B'$ $\quad$ $a\beta m\lambda = C'$ $\quad$ $a\beta m\mu = D'$
$bal\lambda = A''$ $\quad$ $ba\lambda\mu = B''$ $\quad$ $bam\lambda = C''$ $\quad$ $bam\mu = D''$
$b\beta l\lambda = A'''$ $\quad$ $b\beta l\mu = B'''$ $\quad$ $b\beta m\lambda = C'''$ $\quad$ $b\beta m\mu = D'''$,

and the commutant expanded becomes

$$\begin{aligned} &A(B'+C''+C'+B'')\,D''' + (B+C)\,(D'+D'')\,A''' + D(A'+A'')\,(B'''+C''') \\ &-(B+C)\,(A'+A'')\,D''' - A(D'+D'')\,(B'''+C''') - D(B'+C''+C'+B'')\,A'''. \end{aligned}$$

In the foregoing the x's in the several lines were *for the moment* taken identical, in order the more easily to explain the law of formation of the quantities A.

But suppose that they become actually identical for the same line, F then becomes a function of the n^{th} degree in respect to each of p systems of variables, and may be represented symbolically under the form

$$\begin{aligned} &({}^1a^1x + {}^1b^1y + \ldots\ldots {}^1l^1t)^n \times ({}^2a^2x + {}^2b^2x + \ldots\ldots + {}^2l^2t)^n \\ &\qquad\qquad \ldots\ldots \times ({}^pa^px + {}^pb^px + \ldots\ldots + {}^pl^pt)^n. \end{aligned}$$

We may still further limit the generality of the theorem by supposing

$${}^1x = {}^2x = \ldots\ldots {}^px = x,$$
$${}^1y = {}^2y = \ldots\ldots {}^py = y,$$
$$\text{\&c.}$$
$${}^1t = {}^2t = \ldots\ldots {}^pt = t;$$

F then becomes $(ax + by + \text{\&c.} \ldots\ldots + lt)^{np}$.

Accordingly, as many different factors as can be found contained an even number of times in the exponent of the function, so many invariants can be formed immediately from a function of any number of variables m by the method of total commutation.

If one of these factors be called n, the commutant corresponding thereto will be of the order

$$\frac{(n+1)(n+2)\ldots.(n+m-1)}{1.2\ldots.(m-1)}$$

in respect to the coefficients. Thus $m = 2$, so that

$$F = (ax + by)^{np}.$$

The general form of such a commutant will be found by taking $A_1A_2\ldots.A_{n-1}$, the coefficients of the several combinations of x, y in $(ax + by)^n$, from which the numerical coefficients $n, n.\frac{n-1}{2}$, &c. may be rejected, as only introducing a numerical factor into the result; the commutant will therefore be expressed by means of the form

$$a^n;\ a^{n-1}.b;\ a^{n-2}.b^2\ldots.;\ b^n \qquad (1)$$

$$a^n;\ a^{n-1}.b;\ a^{n-2}.b^2\ldots.;\ b^n \qquad (2)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a^n;\ a^{n-1}.b;\ a^{n-2}.b^2\ldots.;\ b^n \qquad (p).$$

If $p = 2$, the compound commutant

$$a^n;\ a^{n-1}.b;\ \ldots.;\ b^n$$

$$a^n;\ a^{n-1}.b;\ \ldots.;\ b^n$$

will easily be seen to be only another form for the catalecticant of $(ax + by)^{2n}$. Thus, let $n = 2$,

$$(ax + by)^4 = Ax^4 + 4Bx^3y + 6Cx^2y^2 + 4Dxy^3 + Ey^4;$$

so that $a^4 = A,\ a^3b = B,\ a^2b^2 = C,\ ab^3 = D,\ b^4 = E.$

The commutant (which is of the form of the matrix to an ordinary determinant, with the exception that the umbræ enter compoundly instead of simply into the several terms separated by the marks of punctuation,) will be

$$a^2;\ ab;\ b^2$$

$$a^2;\ ab;\ b^2:$$

this, written in the six forms

$$\left.\begin{matrix} a^2; & ab; & b^2 \\ a^2; & ab; & b^2 \end{matrix}\right\} \quad \left.\begin{matrix} a^2; & ab; & b^2 \\ a^2; & b^2; & ab \end{matrix}\right\} \quad \left.\begin{matrix} a^2; & ab; & b^2 \\ ab; & a^2; & b^2 \end{matrix}\right\}$$

$$\left.\begin{matrix} a^2; & ab; & b^2 \\ b^2; & ab; & a^2 \end{matrix}\right\} \quad \left.\begin{matrix} a^2; & ab; & b^2 \\ ab; & b^2; & a^2 \end{matrix}\right\} \quad \left.\begin{matrix} a^2; & ab; & b^2 \\ b^2; & a^2; & ab \end{matrix}\right\}$$

gives the expression

$$a^4 \times a^2b^2 \times b^4 - a^4 \times (ab^2)^2 - b^4 \times (a^2b)^2 - (a^2b^2)^2 + 2a^2b \times ab^2 \times a^2b^2;$$

$$\text{i. e. } ACE - AD^2 - EB^2 - C^2 + 2BCD.$$

One important observation may here be made of a fact which otherwise might easily escape attention, which is, that commutants, where the same terms simple or compound are found in all or several of the lines, in general give rise to products, some of them equal and with the same sign, and others equal but with the *contrary* sign.

This last phenomenon does not manifest itself in commutants appertaining to functions of two variables of the two particular and different species which first and most naturally present themselves, viz. where there are only two lines or only two columns*—I believe that it displays itself in every other case of commutantives to functions of two variables. Thus it is that algebraical expressions derived from given functions disguise their symmetry; to make which come to light it becomes necessary to add terms of contrary sign to such expressions. As an example, the reader is invited to develope the cubic invariant of a function of x and y, symbolically expressed by $(ax + by)^8$, where

$$a^8 = A, \quad a^7.b = B \ldots . \ ab^7 = Hb^8 = I,$$

by means of the commutant

$$\begin{matrix} a^2 & ab & b^2 \\ a^2 & ab & b^2 \\ a^2 & ab & b^2 \\ a^2 & ab & b^2. \end{matrix}$$†

* These commutants give respectively the quadrinvariant and the catalecticant, the former of which alone was formerly recognised by Mr. Cayley as a commutant.

† The number of terms resulting from the independent permutation of each of the 3 linear lines is 6^3, that is 216; but the actual result is (using small letters instead of large) $P - Q$, where

$$P = aei + 3ag^2 + 12beh + 3c^2i + 24cf^2 + 24d^2g + 15e^3,$$

$$Q = 4afh + 4bid + 8bgf + 22ceg + 8chd + 36def,$$

so that the effective number of permutations is only 164. The difference

Suppose F to be the general even-degreed function of two variables of the degree $2np$.

Let $$H = \left(\xi \frac{d}{dy} - \eta \frac{d}{dx}\right)^{np} . F + \lambda (x\xi + y\eta)^{np},$$

and express H umbrally under the form

$$(ax + by)^{np} . (a\xi + \beta\eta)^{np}.$$

The commutant

$$\begin{array}{llll} a^n & a^{n-1}.b & \ldots . & b^n \quad (1) \\ a^n & a^{n-1}.b & \ldots . & b^n \quad (2) \\ \vdots & \vdots & \vdots & \vdots \\ a^n & a^{n-1}.b & \ldots . & b^n \quad (p) \end{array}$$

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between this and 216 divided by 216 may be termed the Index of Demolition, which we see in this case is $\frac{5^5}{2^{16}}$ or $\frac{1^3}{5^4}$; that is, somewhat less than $\frac{1}{4}$. For the cubic invariant of the function of the fourth degree this index is zero, all the permutations being effective. If we take the cubic invariant of the function $ax'^3 + 12bx''y + 6bcx'y^3 +$ &c. $+ my'^3$ under the form $P - Q$, we shall find

$$P = 6ahl + 10ajj + 6bfm + 54bhk + 54cfl + 155cii + 10ddm + 430dgj + 155eek + 520ehh + 520ffl + 280ggg,$$

$$Q = agm + 15aik + 30bgl + 50bij + 15cem + 4cgk + 150chj + 30del + 210dfk + 250dhi + 230efj + 555egl + 660fgh.$$

The number of terms in P and Q is of course the same, and will be found to be 2200 for each; so that out of the 6^5, that is 7776 permutations of the 5 lower rows, only 4400 are effective, and the index of demolition becomes $\frac{3300}{7776}$, that is $\frac{275}{648}$, or rather greater than $\frac{5}{12}$. The Index of Demolition thus goes on constantly increasing as the degree of the function rises; probably (?) it converges either towards $\frac{1}{2}$ or else towards unity. In arranging the terms it will be found most convenient to adopt, as I have done above, the dictionary method of sequence. The computations are greatly facilitated by the circumstance of the effect of any arrangement of each of the 5 lower lines not being altered when these *lines* are permuted with one another; this gives rise to the subdivision of the 7776 permutations into groups as follows: 6 of 120 identical terms, 60 of 60, 36 of 20, 60 of 30, 24 of 20, 30 of 10, 30 of 5, and 6 of 1. So that the total number of permutational arrangements to be constructed is only 252. Other methods of abridging the labour will readily suggest themselves to the practical computer. The total number of the *groups* of terms is of course always known *à priori*, and, for instance, in the case before us, must be equal to the number of ways in which $\frac{1}{2}(12 \times 3)$, that is the number 18, can be divided into 3 parts, none of which is to exceed the number 12, that is 25; for the cubic invariant of the function of the eighth degree of two variables it is the number of ways in which 12 can be divided into 3 parts, of which none shall exceed 8, and so forth, zeros being always understood to be admissible; and of course in general for an invariant of the order r to a function of the degree n of i variables, the number of distinct terms is in general the number of ways in which $\frac{nr}{r}$ can be divided into r parts, of which none shall exceed n, subject however always to the possibility in particular cases of a diminution in consequence of some of the groups assuming zero for their coefficient.

$$\begin{array}{llll} a^n & a^{n-1}.\beta \ldots . & \beta^n & (1) \\ a^n & a^{n-1}.\beta \ldots . & \beta^n & (2) \\ \vdots & \vdots \quad \vdots & \vdots & \\ a^n & a^{n-1}.\beta \ldots . & \beta^n & (p) \end{array}$$

will be a function of λ, and all the several coefficients will be invariants of F.*

When $p = 1$ we obtain the Λ given in the preceding section, and originally published by me in the *Philosophical Magazine* for the month of November 1851. The Λ obtained on this supposition has for its coefficients a series of independent invariants, commencing with the catalecticant and closing with the quadratic invariant. When p has any other value, we observe a similar series commencing with a commutantive invariant of a lower order than the catalecticant, but always closing with the quadratic invariant. Thus (*ex. gr.*), when $2np = 8$, we may obtain by the preceding theorem three different quadratic functions; one giving the invariants of the orders 5, 4, 3, 2, the second those of the orders 3, 2, the third the invariant of the order 2.

In this case the invariants of the same order given by the different Λ's are the same to numerical factors *près*. Whether this is always necessarily the case is a point reserved for further examination.

The commutants applied in the preceding theorems have been called by me total commutants, because the total of each line of umbræ is permuted in every possible manner. If the lines be divided into segments, and the permutation be local for each segment instead of extending itself over the whole line, we then arrive at the notion of partial commutants, to which I have also (in concert with Mr. Cayley) given the distinctive name of Intermutants. In order to find the invariants of functions of odd degrees, the theory of total commutants requires the process of commutation to be applied, not immediately to the coefficients of the proposed function, but to some derived concomitant form. I became early sensible of this imperfection, and stated to the friend

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* By substituting the symbols $\frac{d}{dx}, \frac{d}{dy}$, &c. in place of the *umbræ a, b*, &c., the theorem is easily stated for covariants generally. But in applying the commutantive method to obtain covariants, or rather in the statement of the results flowing from each application, it is never necessary to go beyond the case of invariants, because the commutantive covariants of any given homogeneous function are always identical with commutantive invariants of emanants of the same function.

above named, to whom I had previously imparted my general method of total commutation, my conviction of the existence of a qualified or restricted method of permutation, whereby the invariants of the cubic function, for instance, of two and of three letters would admit, without the aid of a derived form of being represented. Many months ago, when I was engaged in this important research, and had made some considerable steps towards the representation of the invariant, *i. e.* the discriminant of the cubic function of x and y under the form of a single permutant, I was surprised by a note from the friend above alluded to, announcing that he had succeeded in fixing the form of the permutant of which I was at that moment in search. It is with no intention of complaining of this interference on the part of one to whose example and conversation I feel so deeply indebted, (and the undisputed author of the theory of Invariants,) that I may be permitted to say that, independent of the intervention of this communication, I must inevitably have succeeded in shaping my method so as to furnish the form in question; and that with the greater certainty, after my theory of commutants had furnished me with the precedent of permutable forms giving rise to terms identical in value but affected with contrary signs. As I have understood that Mr. Cayley is likely to develope this part of the subject in the present number of the *Journal*, it will be the less necessary for me to enter at any length into the theory of partial commutants on the present occasion.

The method of partial commutation is a simple but most important corollary from that of total commutation hereinbefore explained. To fix the ideas, conceive a class of p congredient systems, and that there are qr such classes perfectly independent. Proceed to divide these qr classes in any manner whatever into r sets, each containing q classes; and form the symbol of the total commutant corresponding to each such set. Now let these commutants be placed side by side against one another, and transpose the terms in each compound line thus formed once for all, but in any arbitrary manner. Then permute in every possible way all those symbols in each line, *inter se*, which belong to the same class, and operate with the symbols thus produced by reading off the vertical columns and attending to the rule of the + and − signs, as in the case of a total commutant; the result will be a commutant of the form operated upon. For instance, let $p = 1$, $q = 3$, $r = 2$, and let the number of variables in each system be 2. Form the commutant operators

$$\begin{array}{cc|cc} \frac{d}{dx} & \frac{d}{dy} & \frac{d}{d\xi} & \frac{d}{d\eta} \\ \frac{d}{dp} & \frac{d}{dt} & \frac{d}{d\phi} & \frac{d}{d\theta} \\ \frac{d}{dr} & \frac{d}{ds} & \frac{d}{d\rho} & \frac{d}{d\sigma} \end{array}$$

Interchange *in any manner* but *once for all* the symbols in each line, as thus:

$$\begin{array}{cccc} \frac{d}{dx} & \frac{d}{dy} & \frac{d}{d\xi} & \frac{d}{d\eta} \\ \frac{d}{d\phi} & \frac{d}{dp} & \frac{d}{dt} & \frac{d}{d\theta} \\ \frac{d}{ds} & \frac{d}{d\rho} & \frac{d}{dr} & \frac{d}{d\sigma} \end{array}$$

Now permute, *inter se*, the variables of each system, as

$$\frac{d}{dx}, \frac{d}{dy}; \quad \frac{d}{dp}, \frac{d}{dt}, \text{ \&c.;}$$

the total number of the operative forms resulting will be $(1.2)^s$, and the sum of the $(1.2)^6$ quantities, half positive and half negative, formed after the type of

$$\Sigma \left\{ \begin{array}{l} \frac{d}{dx} \cdot \frac{d}{d\phi} \cdot \frac{d}{ds} \; U \times \frac{d}{dy} \cdot \frac{d}{dp} \cdot \frac{d}{d\rho} \cdot U \\ \times \frac{d}{d\xi} \; \frac{d}{dt} \; \frac{d}{dz} \; U \times \frac{d}{d\eta} \cdot \frac{d}{dt} \cdot \frac{d}{d\sigma} \cdot U \end{array} \right\},$$

U being supposed to be a function homogeneous in

$$x, y; \; \xi, \eta; \; p, t; \; \phi, \theta; \; r, s; \; \rho, \sigma,$$

will be a covariant of U.

The proof of the truth of this proposition is contained in what is shewn in the Notes of the Appendix for total commutants, it being only necessary to make the systems which are independent vary consecutively, and then apply the inference to the supposition of their varying simultaneously.

It may be extended to the more general supposition of classes of an unequal number of congredient systems of unequal numbers of variables in each, the only condition apparently required being that the number of distinct terms shall be the same in each line of the final commutantive operator. The important remark to be made is, that in

applying this theorem there is nothing to prevent any of the systems being made *identical;* or, in other words, a given function of one system of variables may be regarded as a function of as many different, although coincident, sets as we may choose to suppose. Thus, suppose

$$U = Ax^2 + 2Bxy + Cy^2,$$

we may take the partial commutant formed of the two total commutant operators

$$\frac{d}{dx}\quad\frac{d}{dy}$$
$$\frac{d}{dx}\quad\frac{d}{dy}$$

combined with itself. If we write them in the same order,

$$\frac{\dot{d}}{dx}\quad\frac{\dot{d}}{dy}\quad\frac{\acute{d}}{dx}\quad\frac{\acute{d}}{dy}$$
$$\frac{\dot{d}}{dx}\quad\frac{\dot{d}}{dy}\quad\frac{\acute{d}}{dx}\quad\frac{\acute{d}}{dy}$$

(where I use the dots and dashes to distinguish those in the same line which are considered as belonging to the same class, and therefore as permutable, *inter se,*) we shall evidently obtain $4\{AC - B^2\}^2$; if we commence with a permutation, so as to have the form of operation

$$\frac{\dot{d}}{dx}\quad\frac{\dot{d}}{dy}\quad\frac{\acute{d}}{dx}\quad\frac{\acute{d}}{dy}$$
$$\frac{\dot{d}}{dx}\quad\frac{\acute{d}}{dy}\quad\frac{\acute{d}}{dx}\quad\frac{\dot{d}}{dy}$$

it will be found that we obtain $2\{AC - B^2\}^2$.

Again, suppose that we have

$$U = Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3.$$

If we write

$$\frac{\dot{d}}{dx}\quad\frac{\dot{d}}{dy}\quad\frac{\acute{d}}{dx}\quad\frac{\acute{d}}{dy}$$
$$\frac{\dot{d}}{dx}\quad\frac{\dot{d}}{dy}\quad\frac{\acute{d}}{dx}\quad\frac{\acute{d}}{dy}$$
$$\frac{\dot{d}}{dx}\quad\frac{\dot{d}}{dy}\quad\frac{\acute{d}}{dx}\quad\frac{\acute{d}}{dy}$$

the value of the commutant would come out zero; but if we make a permutation, and write

$$\begin{matrix} \dot{d}/dx & \dot{d}/dy & \acute{d}/dx & \acute{d}/dy \\ \dot{d}/dx & \dot{d}/dy & \acute{d}/dx & \acute{d}/dy \\ \dot{d}/dx & \acute{d}/dy & \acute{d}/dx & \dot{d}/dy \end{matrix}$$

the operation indicated by the above performed upon U, will give a multiple of the discriminant of U.

In like manner we may represent Aronhold's Sextic Invariant of the form $(x, y, z)^3$ by means of the partial commutant

$$\begin{matrix} \dot{d}/dx & \dot{d}/dy & \dot{d}/dz & \acute{d}/dx & \acute{d}/dy & \acute{d}/dz \\ \acute{d}/dx & \dot{d}/dy & \dot{d}/dz & \dot{d}/dx & \acute{d}/dy & \acute{d}/dz \\ \acute{d}/dx & \acute{d}/dy & \dot{d}/dz & \dot{d}/dx & \dot{d}/dy & \acute{d}/dz \end{matrix}$$

If we make

$$V = \left(\xi' \frac{d}{d\xi} + \eta' \frac{d}{d\eta} + \zeta' \frac{d}{d\zeta}\right) \cdot \left(\xi \frac{d}{dx} + \eta \frac{d}{dy} + \zeta \frac{d}{dz}\right)^2 \cdot (x, y, z)^3,$$

and use H to signify the determinant

$$\begin{matrix} x\,;\ y\,;\ z, \\ \xi\,;\ \eta\,;\ \zeta, \\ \xi'\,;\ \eta'\,;\ \zeta' \end{matrix}$$

which is evidently an universal triple covariant, and make

$$W = V + \lambda H,$$

and apply to W the partial commutantive symbol

$$\begin{matrix} \dot{d}/dx & \dot{d}/dy & \dot{d}/dz & \acute{d}/dx & \acute{d}/dy & \acute{d}/dz \\ \acute{d}/d\xi & \dot{d}/d\eta & \dot{d}/d\zeta & \dot{d}/d\xi & \acute{d}/d\eta & \acute{d}/d\zeta \\ \acute{d}/d\xi' & \acute{d}/d\eta' & \dot{d}/d\zeta' & \dot{d}/d\xi' & \dot{d}/d\eta' & \acute{d}/d\zeta' \end{matrix}$$

we shall obtain a function of λ of which all the odd powers and the second power will disappear, and such that the coefficients of λ^2 and the constant term will be Aronhold's S and T, and the discriminant of the entire function in respect to λ^2 (if not for the distribution assigned to the dots and dashes in the foregoing, at least for some other distribution) may not improbably be the discriminant of the given function $(x, y, z)^3$.

[*To be Continued.*]

NOTES IN APPENDIX.

(1) More generally, in as many ways as the number n can be divided into parts, in so many ways can a given function of one set of variables be as it were *unravelled* so as to furnish concomitant forms.

For instance, the form $ax^3 + 3bx^2y + 3cxy^2 + dy^3$ has for its concomitant

$$aux + buy + bvx + cvy + cwx + dwy,$$

where u, v, w are cogredient with x^2, $2xy$, y^2, and

$$auu'x + buu'y + bu'v'x + bvv'x + cvv'x + cvv'y + cv'v'y + dv'v'y,$$

where u, v; u', v' are cogredient with each other and with x and y; and the proposition in the text may be best derived from this more general theorem by dividing the index into equal parts, forming as many systems as there are such parts, and then identifying the systems so formed.

(2) The following additional example will illustrate the power of this method.

Let $\phi = (x, y, z)^4$ be the general function of the fourth degree. Form by *unravelment* the concomitant form $(u, v, w, p, q, r)^2$ (say P) where u, v, w, p, q, r are congredient with x^2, y^2, z^2, $2zy$, $2xz$, $2yx$.

Again, the universal concomitant $(x\xi + y\eta + z\zeta)^2$ will have for its concomitant

$$u\xi^2 + v\eta^2 + w\zeta^2 + p\eta\zeta + q\zeta\xi + r\xi\eta,$$

where ξ, η, ζ are contragredient to x, y, z. Now take the reciprocal polar of this last form with respect to ξ, η, ζ; that is,

$$\Sigma\,(vw - \tfrac{1}{4}p^2)\,x_1^2 + 2\Sigma\,(\tfrac{1}{4}qr - \tfrac{1}{2}pu)\,y_1z_1 \text{ (say } G),$$

where x_1, y_1, z_1 being contragredient to ξ, η, ζ, will be cogredient with x, y, z. $P + \lambda G$ is a quadratic function of the six variables u, v, w, p, q, r, and its discriminant will give a function of λ of the sixth degree, all of whose even coefficients will be covariants of ϕ. If we replace x_1, y_1, z_1 by x, y, z, these even coefficients will be respectively (understanding that *order* refers to the dimensions *quoad* the coefficients of ϕ and degree to the dimensions *quoad* x, y, z) as follows:

Of order 6	degree	0,
5	"	2,
4	"	4,
3	"	6,
2	"	8,
1	"	10,
0	"	12.

The two last coefficients must evidently be identically zero. It is *possible*

that some of the others may be so too: as regards the one of the third order and sixth degree, this is of the same form as, and may be identical with, the Hessian of ϕ; as regards the one of the fourth order and fourth degree, this may be ϕ itself multiplied by the cubic invariant (which the theory of Section 3 proves to exist) of ϕ. But the covariants of the fifth order and second degree, and of the second order and eighth degree, if they are not identically zero, and if the latter is not ϕ^2 (which a trial or two of some very simple cases will easily establish one way or the other) are probably irreducible forms. The existence of a correlated conic section to a curve of the fourth order, if established, would be particularly interesting, and its geometrical meaning would well deserve being elicited.

(3) If any form (f) of the degree n be written symbolically,

$$(a_1x_1 + a_2x_2 + \ldots\ldots + a_ix_i)^n,$$

where $x_1, x_2 \ldots x_i$ are real but $a_1, a_2 \ldots a_i$ umbral, and if I_r be any invariant of the order r in respect of the real coefficients of (f), it is easily seen by reason of I_r remaining unaltered when $x_1, x_2 \ldots x_i$ become respectively $f_1x_1, f_2x_2 \ldots f_ix_i$, provided that $f_1, f_2 \ldots f_i = 1$, that each term in I_r expressed by means of the umbræ, must contain an equal number of times $a_1, a_2 \ldots a_i$, so that each such term will contain $\frac{nr}{i}$ of each of them, of course differently subdivided and grouped; hence we have the universal condition that $\frac{nr}{i}$ must be an integer; but this is less stringent than the actual condition, which is that $\frac{nr}{i}$ must be an integer of a certain form; for instance, as before observed, when $i = 2$ $\frac{nr}{i}$ must be an *even* integer.

(4) To prove the theorem given in the text for total simple commutants it is only necessary to bear in mind that whenever two columns in any total commutant become identical, the commutant vanishes. To fix the ideas, take the commutant formed of lines similar to $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, written under one another; let there be (r) such lines, the total number of terms will be $(1.2.3)^r$: the 1.2.3 positions of the line written above will correspond to $(1.2.3)^{r-1}$ several groupings of the remaining lines; now when x, y, z undergo a uni-modular linear substitution, $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ will undergo a related substitution not coincident with that of x, y, z, but still uni-modular; let x, y, z change all the other systems remaining fixed, and suppose $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ to become respectively

$$f\frac{d}{dx} + g\frac{d}{dy} + h\frac{d}{dz},$$

$$f'\frac{d}{dx} + g'\frac{d}{dy} + h'\frac{d}{dz},$$

$$f''\frac{d}{dx} + g''\frac{d}{dy} + h''\frac{d}{dz},$$

then each of the $(1.2.3)^{r-1}$ groups of the terms arising from the permutation of $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$ will subdivide into 27 groups, of which we may reject those in which any of the terms $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$ occurs twice or three times; accordingly there will be left only the six effective orders of permutations,

$$f\frac{d}{dx},\ g'\frac{d}{dy},\ h''\frac{d}{dz};\quad f\frac{d}{dx},\ h'\frac{d}{dz},\ g''\frac{d}{dy};\quad \&c.$$

consequently each of the $(1.2.3)^{r-1}$ groups gives rise to 6 times 6 products whose sum will be $\left.\begin{matrix} f;\ g;\ h \\ f';\ g';\ h' \\ f'';\ g'';\ h'' \end{matrix}\right\} \times$ the sum of the 6 products corresponding to the permutations of $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$; and therefore, the transformations being uni-modular, the sum of the products corresponding to the entire $(1.2.3)^r$ permutations remains constant when x, y, z change. In like manner, all the systems may change one after the other, and consequently all of them at the same time without affecting the value of the commutant: and in like manner for the general case. Q. E. D.

(5) The truth of the proposition relative to compound commutants and the mode of the demonstration will be apparent from the subjoined example.

Let the function be supposed to be

$$(ax + by)\,(a'x' + b'y')\,(a\xi + \beta\eta)\,(a'\xi' + \beta'\eta'),$$

where $x, y; x', y'$ are cogredient and $\xi, \eta; \xi', \eta'$ cogredient; the a, b, a, β, &c. are of course mere umbræ. Now take the compound commutant

$$aa';\quad ab' + a'b;\quad bb',$$
$$a\alpha';\quad a\beta' + a'\beta;\quad \beta\beta'.$$

Let $x, y; x', y'$ undergo a linear substitution, and, accordingly,

$$\begin{matrix} \text{let } a & \text{become } fa + gb, \\ a' & \text{“} \quad fa' + gb', \\ b & \text{“} \quad ha + kb, \\ b' & \text{“} \quad ha' + kb', \end{matrix}$$

f, g, h, k being of course actual and not umbral; then the above commutant will be easily seen to decompose into 6 others, which will be equal to the original commutant multiplied by the determinant

$$\begin{matrix} f^2; & 2fg; & g^2, \\ fh; & fk + gh; & gk, \\ h^2; & 2hk; & k^2, \end{matrix}$$

which is equal to $(fk - gh)^3$, i. e. $= 1$.

And so in general, which shews, as in the preceding note, that all the classes of congredient systems may be transformed successively one after the other, and therefore simultaneously, without altering the value of the commutant.

(6) In the last May Number of the *Journal*, Mr. Boole, to whose modest labours the subject is perhaps at least as much indebted as to any one other writer, has given a theorem,* (14) p. 94, the excellent idea contained in which there is no difficulty in shaping so as to render it generalizable by aid of the theory of contravariants. It may be regarded in some sort a pendant or reciprocal to the Eisenstein-Hermite theorem, presented by me under a wider aspect in the First Section of this paper.

Let $\phi(x, y...z)$ have any contravariant $\theta(x, y...z)$; then will

$$\phi\left(\frac{d}{dx}, \frac{d}{dy} \ldots\ldots \frac{d}{dz}\right) . \theta(x, y...z)$$

be a contravariant of ϕ. For orthogonal transformations the terms contravariant and covariant coincide, and the theorem for this case appears to have been known to Mr. Boole, see (15), same page. More generally, if ψ and θ be any two concomitants of ϕ, the algebraical product $\psi.\theta$ will also be a concomitant of ϕ, provided that the systems of variables in ψ and θ have all distinct names, or that those which bear the same names are cogredient with another. If this proviso does not hold good, the product in question will evidently be no longer a concomitant of ϕ. Let however Ψ denote what ψ becomes, and ϑ what θ becomes, when in place of the variables $x, y.....z$ of every two contragredient synonymous systems in ψ and θ we write $\frac{d}{dx}, \frac{d}{dy} ..... \frac{d}{dz}$, then will $\vartheta.\psi$ and $\Psi.\theta$ be each of them concomitants of ϕ, the synonymous systems becoming cogredient with ψ in the one case and with θ in the other.

(7) There is one principle of *paramount* importance which has not been touched upon in the preceding pages, which I am very far from supposing to exhaust the fundamental conceptions of the subject, (indeed, not to name other points of enquiry, I have reason to suppose that the idea of contragredience itself admits of indefinite extension through the medium of the reciprocal properties of commutants; the particular kind of contragredience hereinbefore considered having reference to the reciprocal properties of ordinary determinants only).

The principle now in question consists in introducing the idea of *continuous* or *infinitessimal* variation into the theory. To fix the ideas, suppose C to be a function of the coefficients of $\phi(x, y, z)$, such that it remains unaltered when x, y, z become respectively fx, gy, hz, provided that $fgh = 1$. Next, suppose that C does not alter when x becomes $x + ey + \varepsilon z$, when e and ε are indefinitely small: it is easily and obviously

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* Mr. Boole applied his theorem to obtain the cubic invariant of $(x, y)^4$, say $\phi(x, y)$, by operating upon its Hessian with $\phi\left(\frac{d}{dy} - \frac{d}{dx}\right)$. More generally, when $\phi(x, y) = (x, y)^{2n}$, the catalecticant of the antepenultimate emanant of ϕ is also of the degree $2r$; and this, when operated upon by $\phi\left(\frac{d}{dy}, -\frac{d}{dx}\right)$, will give an invariant of the order $n + 1$, which is probably identical with the catalecticant of ϕ itself. There exists a most interesting transformation of the catalecticant of any emanant of a function of any degree in x, y, whether even or odd, under the form of a determinant some of the lines of which contain combinations only of x and y, without any of the coefficients, and all the rest the coefficients only of the given function without x or y. The Hessian being the catalecticant of the second emanant is of course included within this statement.

demonstrable that if this be true for e and ϵ indefinitely small, it must be true for *all* values of e and ϵ. Again, suppose that C alters neither when x receives such an infinitesimal increment, y and z remaining constant, nor when y nor z separately receive corresponding increments, z, x and x, y in the respective cases remaining constant, it then follows from what has been stated above that this remains true for finite increments to x or y or z separately, and hence it may easily be shewn that C will remain constant for any *concurrent* linear transformations of x, y, z, when the modulus is unity. This all-important principle enables us at once to fix the form of the symmetrical functions of the roots of $\phi\left(\frac{x}{y}, 1\right)$ which represent invariants of $\phi(x, y)$ when the coefficient of the highest power of x is made unity. It also *instantaneously* gives the necessary and *sufficient* conditions to which an invariant of any given order of any homogeneous function whatever is subject, and thereby reduces the problem of discovering invariants to a definite form. But as these conditions coincide with those which have been stated to me as derived from other considerations by the gentleman whose labours in this department are *concomitant* with my own, I feel myself bound to abstain from pressing my conclusions until he has given his results to the press.

(8) By aid of the general principle enunciated in Note (6) above, we can easily obtain Aronhold's S and T. Let U be the given cubic function of x, y, z, and let $G(x, y, z; \xi, \eta, \zeta)$ be the polar reciprocal in respect to ξ, η, ζ of $\left(\xi\frac{d}{dx} + \eta\frac{d}{dy} + \zeta\frac{d}{dz}\right)^2 U$, then $G(\xi, \eta, \zeta; x, y, z)$ as well as the former G will be a concomitant to U, but the homonymous systems of variables in the two G's will be contragredient; and, accordingly, $G\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}; \xi, \eta, \zeta\right).G(\xi, \eta, \zeta; x, y, z)$ will be a concomitant to U; this concomitant is readily seen to be an invariant of the fourth order; that is, Aronhold's S. Again, from S, by means of the Eisenstein-Hermite theorem, we may derive a form $K(x, y, z)$ of the third degree in x, y, z, and whose coefficients will be of three dimensions; and, accordingly, if the Hessian of U be called $H(U)$,

$$K\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right).H(U)$$

will be a Sextic Invariant of U, that is, Aronhold's T.

CORRECTION OF THE POSTSCRIPT TO THE PAPER ON PERMUTANTS.

By Arthur Cayley.

Mr. Sylvester has represented to me that the paragraph relating to his communications conveys an erroneous idea of the nature, purport, and extent of such communications; I have, in fact, in the paragraph in question, singled out what immediately suggested to me the expression of the function $6abcd + 3b^2c^2 - 4ac^3 - 4b^3d - a^2d^2$ as a partial com-

mutant or intermutant, but I agree that a fuller reference ought to have been made to Mr. Sylvester's results, and that the paragraph in question would more properly have stood as follows:

"Under these circumstances Mr. Sylvester communicated to me a series of formal statements, not only oral but in writing, to the effect that he had discovered a permutation method of obtaining as many invariants—viz. commutantive invariants—by direct inspection from a function of any degree of any number of letters as the index of the degree contains even factors; one of these commutantive invariants being in fact the function $ace + 2bcd - ae^2 - bd^2 - c^3$, expressible, according to Mr. Sylvester's notation, by $\begin{pmatrix} a^2, & ab, & b^2 \\ a^2, & ab, & b^2 \end{pmatrix}$; and, according to the notation of my memoir in the *Camb. Phil. Trans.*, supposing $00 = a$, $01 = 10 = b$, $02 = 11 = 20 = c$, &c. by $\begin{vmatrix} 00 \\ 11 \\ 22 \end{vmatrix}$."

Mr. Sylvester and I shall, I have no doubt, be able to agree to a joint statement of any further correction or explanation which may be required.

Jan. 27, 1852.

ON DEMONSTRATIONS OF THE BINOMIAL THEOREM.

By Homersham Cox, B.A.

The Binomial Theorem fails arithmetically when it expands a finite power of a binomial in an infinite divergent series. For instance, if $(\frac{1}{3})^2$ be expressed by the expansion of $(1-4)^{-2}$ by this theorem,

$$\tfrac{1}{9} = 1 + 2.4 + 3.4^2 + 4.4^3 + \ldots \text{ } \textit{ad infinitum.}$$

The second side of this equation is divergent and its arithmetical sum infinite. We have therefore $\frac{1}{9} = \infty$, an obvious absurdity if the symbol = designate arithmetical equality. But it is said that the symbol = here designates *symbolical equivalence.* The truth of this assertion depends on the definition of this phrase, and without doubt many arbitrary definitions might be given, in accordance with which the Binomial Theorem might be considered to hold for divergent series. But if symbolical equivalence be ever interpreted to include arithmetical equivalence, it seems

clear from the foregoing consideration, that the definition ought to admit that interpretation for convergent, and exclude it for divergent series expanded by the Binomial Theorem.

In the best known demonstrations of the theorem the meaning of the symbol = is certainly meant to include *always* arithmetical equality, and yet the reasoning of these demonstrations in no way excludes expansion by divergent series. As they therefore necessarily lead to the absurdity that a finite arithmetical quantity may equal infinity, it appears certain that they contain incorrect steps.

In Euler's proof $f(m)$ being defined by the equation

$$f(m) = 1 + mx + \frac{m \,.\, m-1}{1.2} x^2 + \frac{m \,.\, m-1 \,.\, m-2}{1.2.3} x^3 + \text{\&c. } ad\ inf.$$

it is inferred from the case when m and n are positive integers that $f(m).f(n) = f(m+n)$ when m and n are positive or negative integral or fractional. Now in the series for $f(m+n)$ the $(r+1)^{\text{th}}$ term is

$$\frac{n+m \,.\, n+m-1 \,.\, n+m-2 \ldots . n+m-r+1}{1.2.3 \ldots . r} x^r,$$

which when r is infinite takes the form

$$\frac{n+m \,.\, n+m-1 \,.\, n+m-2 \ldots . n+m-\infty+1}{1.2.3 \ldots . \infty} x^\infty,$$

an indeterminate quantity.

But if m or n be fractional or negative, the series in which they are involved are continued *ad infinitum*, and then, therefore, r becomes infinite. In order therefore to find the value of $f(m+n)$ from its expansion in such cases, the sum of terms involving indeterminate quantities has to be evaluated.

This consideration does not appear to be regarded in Euler's demonstration, which, as will be shewn, tacitly assumes that *the sum of those terms of an infinite series which contain a zero factor is necessarily zero*, an assumption frequently erroneous.

The demonstration for negative indices is made to depend on the assumption that $f(n-n) = 1$. But the development of $f(n-n)$ is

$$1 + n-n \,.\, x + \frac{n-n \,.\, n-n-1}{1.2} x^2 + \ldots . + \frac{n-n \,.\, n-n-1 \ldots . n-n-r+1}{1.2 \ldots . r} x^r$$

$$+ \text{\&c. } ad\ inf.,$$

and this series cannot = 1 unless the sum of all the terms in it following 1 be zero. Now though each of these terms is zero when r is finite, they when r is infinite take the form

$$\frac{0. - 1. - 2. \ldots, (- \infty + 1)}{1.2.3. \ldots \infty} x^{\infty},$$

an indeterminate quantity not zero unless x be a fraction.

Similarly the demonstration for fractional indices depends on the assumption that

$$f\left(\frac{h}{k} + \frac{h}{k} + \text{\&c. to } k \text{ terms}\right) = (1 + x)^h,$$

h and k being positive integers. The development of the first side of the equation is an infinite series with terms having zero and infinite factors. The objection against neglect of these terms is the same as in the demonstration for negative indices.

It may be observed that the demonstration for fractional indices assumes also that any fraction may be expressed by the ratio of two integers. This assumption excludes incommensurable fractions, but for such indices the proof might probably be completed by considerations somewhat resembling those by which Duchayla extends his proof of the parallelogram of forces to the case of incommensurable components.

In Lund's edition of Wood's *Algebra* (Cambridge, 1848), Euler's proof is dismissed to an Appendix to make way for another by the Rev. J. Griffith, introduced in the following terms: "This proof is new and appears to possess peculiar merit.... Any one who still retains an affection for Euler's proof will find it in Note 3." The question of transference of affection to the new demonstration need not be examined here, as precisely the same objections apply to it as to Euler's, from which it differs in no other respect whatever, than in obtaining laboriously by actual multiplication the form of the series which is the product of two series of the form $f(m)$ above.

In Hutton's *Mathematical Tracts* (London, 1812) an account is given of several demonstrations of the Binomial Theorem. The following is a brief abstract of this account. James Bernouilli, in his *Ars Conjectandi*, demonstrated the theorem for positive integral indices. John Stewart, in his *Commentary on Newton's Quadrature of Curves*, copied Bernouilli's demonstration, and added a demonstration for fractional exponents by the principles of fluxions. De Moivre, in the

Philosophical Transactions, No. 230 for 1697, demonstrated the multinomial theorem and the binomial theorem as a particular case of it, by the principles of combinations and permutations. Landen in his *Discourses concerning the Residual Analysis*, 1758, and *Residual Analysis*, 1764, proves the theorem by assuming that a binomial raised to a fractional power is equal to a series ascending by integral powers of one term of the binomial, and equating coefficients. He compares this investigation with that by fluxions in which the above series is differentiated. Hales, in his *Analysis Equationum*, gives an account of this investigation. Hutton, in the above work cited, gives a proof of his own by assuming a series similar to that just described, and equating coefficients. On reference to Stewart's *Commentary*, it appears that he first shews that if x be a flowing quantity the fluxion of x^n is $nx^{n-1}x$; he then assumes a series for the expansion of the binomial raised to any power, and differentiates.

In the *Penny Cyclopædia* (BINOMIAL THEOREM) is a notice of a proof by Messrs. Swinburne and Tylecote (*Cambridge*, 1827), which is stated to be, if the details be correct, "of a logical character, but far above the student."

None of the proofs mentioned by Dr. Hutton, nor his own, appears to regard the convergency of the series demonstrated. The same neglect of convergency appears in most demonstrations of the principle of equation of coefficients. But supposing it admitted that the meaning of the symbol = is always to include the signification *arithmetical equality*, an exact proof of the Binomial Theorem must certainly have regard to the convergency of the series it involves. Now the investigation, whether the sum of a series have a finite limit which cannot be exceeded however great the number of terms—what is this but a question of the doctrine of limits? It seems certain therefore that this doctrine must be involved at some stage or other of the required exact proof, and that any attempt to escape from this necessity by algebraical artifices merely evades, without avoiding the real issue of the question. If however the principles of limits be necessarily involved, they surely ought to be openly avowed and distinctly applied. So that the desideratum in demonstrating the Binomial Theorem seems to resolve itself into the explanation and application of the parts of the doctrine of limits which the nature of the subject requires, in the form most readily intelligible to the algebraical student. This object might probably be effected by methods much

better than the following, which is offered merely on account of its brevity and as an illustration of the views here taken.

Proof of the Binomial Theorem.

(1). Let m be any finite positive integer. It may be ascertained by the multiplication of $b + k$ by itself m times that $(b+k)^m = b^m + mb^{m-1}k +$ *terms multiplied by the second and higher powers of* k, and from the manner in which this result is obtained, it will easily be seen to be true for all positive integral values of m.

(2). Let the *limit* of a quantity involving k be defined to be the value from which that quantity differs by quantities which decrease indefinitely as k is diminished. For instance, it appears from the above that $\frac{(b+k)^m - b^m}{k} = mb^{m-1} +$ *terms multiplied by the first and higher powers of* k. Now these terms decrease indefinitely as k is diminished; therefore "limit" of $\frac{(b+k)^m - b^m}{k} = mb^{m-1}$. This result can be extended algebraically to all real values of m.

(3). Let X be any quantity involving x, such that when $x = 0$ or h, X is zero but has different values for intermediate values of x. Of these other values of X there must be one at least which is not exceeded in positive magnitude by any other of them. Let this be X_1 corresponding to a value x_1. Then if x have either a large value $x_1 + k$ or smaller $x_1 - k$, the corresponding values X_2 and X_3 do not exceed X_1; or $X_2 - X_1$ and $X_3 - X_1$ have the same sign; therefore $\frac{X_2 - X_1}{k}$ and $\frac{X_3 - X_1}{-k}$ have different signs. Let both these quantities have the same "limit" L, from which, therefore, by the definition, they differ by differences which decrease indefinitely as k is diminished. If then L had any magnitude, these differences might be taken less than it; and then $\frac{X_2 - X_1}{k}$ and $\frac{X_3 - X_1}{-k}$ would have the same sign as L. But it has been shewn that they must have contrary signs; therefore L must not be of any magnitude or must = 0.

(4). Let $X =$

$$(a+x)^n - \left(a^n + na^{n-1}x + \frac{n.n-1}{1.2}a^{n-2}x^2 + \frac{n.n-1.n-2}{1.2.3}a^{n-3}x^3 + \ldots + Ax^r\right) \ldots \text{(I)},$$

r being a positive integer and A such a quantity not involving

x that $X=0$ when $x=h$. Also $X=0$ when $x=0$. When $x=x_1$,

$$\frac{X_2-X_1}{k}=\frac{(a+x+k)^n-(a+x)^n}{k}-\left\{na^{n-1}+\frac{n.n-1}{1.2}a^{n-2}\frac{(x+k)^2-x^2}{k}\right.$$
$$\left.+\ldots+A\frac{(x+k)^r-x^r}{k}\right\}.$$

Giving m successive values in (2) the limit of this is found to be

$$n(a+x)^{n-1}-n\left(a^{n-1}+n-1\,.\,a^{n-2}x+\frac{n-1\,.n-2}{1.2}a^{n-3}x^2+\ldots\right)+rAx^{r-1}\ldots(\text{II}).$$

By writing $-k$ for k, it may easily be seen that $\frac{X_3-X_1}{-k}$ has the same limit, which by (3) = 0 when $x=x_1$. But (II) is also zero when $x=0$. Therefore let the operation by which (II) was obtained from (I) be repeated on (II) and on the successive results from it, each of which becomes zero under like conditions. After r such operations we have finally $0=n.n-1\ldots n-r+1\,.\,(a+x)^{n-r}-1.2\ldots r.A$ for some value of x between 0 and h. Call this value θh where θ is some proper fraction. Then $1.2\ldots r.A=n.n-1\ldots n-r+1(a+\theta h)^{n-r}$. Substituting from this in (1) and putting $X=0$ when $x=h$,

$$(a+h)^n=a^n+na^{n-1}h+\frac{n.n-1}{1.2}a^{n-2}h^2+\ldots+\frac{n.n-1\ldots n-r+1}{1.2\ldots r}(a+\theta h)^{n-r}h^r.$$

If the last term of this series vanish when r is sufficiently large,

$$(a+h)^n=a^n+na^{n-1}h+\frac{n.n-1}{1.2}a^{n-2}h^2+\ldots\text{ to }0,$$

which is the Binomial Theorem.

Cambridge, October 1851.

MATHEMATICAL NOTES.

I.—Solution of a Functional Equation.

To solve the functional equation

$$\int_{-\infty}^{+\infty}\phi_m x\phi_n(a-x)\,dx=\phi_{m+n}a,$$

provided $\phi x=\phi(-x)$.

Let $\int_{-\infty}^{+\infty}\phi a\cos az\,da=\psi z$. Then

$$\psi_{m+n}z=\int_{-\infty}^{+\infty}da\int_{-\infty}^{+\infty}dx\,\phi_m x\phi_n(a-n)\cos az.$$

Write for a, $a + x$, then $\cos az$ becomes

$$\cos az \cos xz - \sin az \sin xz,$$

and as the sines change sign with their arcs, we get

$$\psi_{m+n} z = \int_{-\infty}^{+\infty} da \int_{-\infty}^{+\infty} dx \, \phi_m x \, \phi_n a \cos az \cos xz.$$

Or $$\psi_{m+n} z = \psi_m z \, \psi_n z,$$

whence $$\psi_m z = \chi z^m,$$

χ being independent of m, and therefore, by Fourier's theorem,

$$\phi_m x = \frac{1}{\pi} \int_0^{\infty} \chi z^m \cos xz \, dz.$$

The required solution. R. L. E.

II.—Note on a Question in the Senate-House Papers for 1852.

"The sections of a surface of the second order made by parallel planes are similar and similarly situated conics. Determine the ratio of the magnitudes of the sections made by two given planes."

The equations of the surface of the second order and of the parallel planes being assumed to be

$$lx^2 + my^2 + nz^2 - k = 0,$$

$$ax + \beta y + \gamma z - \delta = 0;$$

then if

$$x = s\xi + a, \quad y = s\eta + b, \quad z = s\zeta + c,$$

and the constants s, a, b, c be properly determined, the two equations will take the form

$$l\xi^2 + m\eta^2 + n\zeta^2 - k = 0,$$

$$\alpha\xi + \beta\eta + \gamma\zeta = 0;$$

which shews that the section made by the plane $\alpha x + \beta\eta + \gamma z - \delta = 0$, is a curve similar and similarly situated to the section made by the parallel plane $\alpha x + \beta y + \gamma z = 0$, and each section is of course a conic. The values of a, b, c, s are

$$a = \frac{\alpha\delta}{Kl}, \quad b = \frac{\beta\delta}{Km}, \quad c = \frac{\gamma\delta}{Kn}, \quad s^2 = 1 - \frac{\delta^2}{kK},$$

where $$K = \frac{\alpha^2}{l} + \frac{\beta^2}{m} + \frac{\gamma^2}{n}.$$

And the quantity s represents the ratio of the magnitudes of the sections made by the two parallel planes. A. C.

ON THE FAMILY OF THE WAVE-SURFACE.

By William Walton.

The Wave-surface in biaxal crystals may be regarded in a light under which, as far as I am aware, it has not been considered, viz. as a member of a family of surfaces connected together by a common principle of generation. I propose in fact to shew that it may be traced out by the skew movement of a generator of double curvature (the intersection of two cylinders of the second order, the axes of which cut each other at right angles), which is subjected to pass through three concentric circular directors in planes at right angles to each other, and then to investigate the partial differential equation of the family of surfaces which are the loci of the same generator regulated by any directors whatever.

(A). To find the locus of the generator

$$\left.\begin{aligned} \frac{y^2}{\mu} - \frac{z^2}{\nu} &= b^2 - c^2 \\ \frac{z^2}{\nu} - \frac{x^2}{\lambda} &= c^2 - a^2 \\ \frac{x^2}{\lambda} - \frac{y^2}{\mu} &= a^2 - b^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (1),$$

λ, μ, ν, being arbitrary parameters, which is constrained always to pass through the three directors

$$\left.\begin{aligned} x &= 0 \\ y^2 + z^2 &= a^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (2),$$

$$\left.\begin{aligned} y &= 0 \\ z^2 + x^2 &= b^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (3),$$

$$\left.\begin{aligned} z &= 0 \\ x^2 + y^2 &= c^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (4).$$

Since the curve (1) passes through each of the curves (2), (3), (4), we easily obtain the three respective equations

$$\begin{aligned} \lambda a^2 + \mu b^2 + \nu c^2 &= (1 + \lambda + \mu + \nu)\, a^2 \\ &= (1 + \lambda + \mu + \nu)\, b^2 \\ &= (1 + \lambda + \mu + \nu)\, c^2, \end{aligned}$$

which are evidently equivalent to the two following:

$$1 + \lambda + \mu + \nu = 0 \quad \ldots\ldots\ldots\ldots (5),$$

$$\lambda a^2 + \mu b^2 + \nu c^2 = 0 \quad \ldots\ldots\ldots\ldots (6).$$

From the equations (1) it is plain that, r being some quantity,

$$\left.\begin{aligned} \frac{x^2}{\lambda} &= a^2 - r^2 \\ \frac{y^2}{\mu} &= b^2 - r^2 \\ \frac{z^2}{\nu} &= c^2 - r^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (7).$$

Hence $x^2 + y^2 + z^2 = \lambda a^2 + \mu b^2 + \nu c^2 - r^2 (\lambda + \mu + \nu)$

$= r^2$, by (5) and (6).

From (5) and (7), we see that

$$\frac{x^2}{r^2 - a^2} + \frac{y^2}{r^2 - b^2} + \frac{z^2}{r^2 - c^2} = 1,$$

which is the equation to the wave-surface, the required locus.

(B). Suppose that $\mu = 0$. Then, from (1), we see that the equations to the corresponding generator are reduced to

$$y = 0,$$

$$\frac{z^2}{\nu} - \frac{x^2}{\lambda} = c^2 - a^2,$$

the equation connecting the variables λ and ν being

$$1 + \lambda + \nu = 0,$$

$$\lambda a^2 + \nu c^2 = 0.$$

From the last two equations we have

$$\nu(c^2 - a^2) = a^2, \quad \lambda(c^2 - a^2) = -c^2,$$

and therefore the generator is an ellipse defined by the equations

$$y = 0,$$

$$\frac{z^2}{a^2} + \frac{x^2}{c^2} = 1.$$

The intersections of this generator with the director (3) are the singular points of the wave-surface.

It may easily be proved that this is the only one of the generators which passes through a singular point.

Suppose, in fact, that a generator, for which μ is not equal to zero, passes through such a point. Then, at the point in question, as we see from (1),

$$\frac{z^2}{\nu} = c^2 - b^2, \quad \frac{x^2}{\lambda} = a^2 - b^2,$$

and therefore

$$\lambda (a^2 - b^2) z^2 + \nu (b^2 - c^2) x^2 = 0.$$

But, at a singular point,

$$c^2 (a^2 - b^2) z^2 - a^2 (b^2 - c^2) x^2 = 0.$$

Hence we see that

$$\lambda a^2 + \nu c^2 = 0,$$

and therefore, by (6), $\mu = 0$, which is contrary to the hypothesis.

(C). Suppose the generator (1) to intersect a consecutive generator. Then, at the point of intersection,

$$\frac{x^2}{\lambda^2} d\lambda = \frac{y^2}{\mu^2} d\mu = \frac{z^2}{\nu^2} d\nu.$$

But, by (5) and (6),

$$d\lambda + d\mu + d\nu = 0,$$

$$a^2 d\lambda + b^2 d\mu + c^2 d\nu = 0:$$

hence

$$\frac{\lambda^2}{x^2} + \frac{\mu^2}{y^2} + \frac{\nu^2}{z^2} = 0,$$

$$\frac{\lambda^2 a^2}{x^2} + \frac{\mu^2 b^2}{y^2} + \frac{\nu^2 c^2}{z^2} = 0,$$

and therefore $\lambda = 0$, $\mu = 0$, $\nu = 0$, values of λ, μ, ν, which are incompatible with the equation (5). Thus we see that consecutive generators do not intersect, or that the generation of the surface is effected by a skew movement.

(D). The generator (1), moving in accordance with the conditions (5) and (6), is evidently a particular instance of a generator, defined by the same equations, which moves in accordance with two general conditions

$$\left.\begin{aligned} \phi(\lambda, \mu, \nu) &= 0 \\ \chi(\lambda, \mu, \nu) &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (8).$$

Thus the locus of (1), the parameters of which are connected by the conditions (8), constitutes a family of surfaces of which the wave-surface is a particular individual.

(E). The equations (1), if x, y, z, be now taken to stand in place of x^2, y^2, z^2, respectively, become

$$\frac{y}{\mu} - \frac{z}{\nu} = b^2 - c^2,$$

$$\frac{z}{\nu} - \frac{x}{\lambda} = c^2 - a^2,$$

$$\frac{x}{\lambda} - \frac{y}{\mu} = a^2 - b^2.$$

Differentiating these equations, we have

$$\frac{dx}{\lambda} = \frac{dy}{\mu} = \frac{dz}{\nu} \quad \ldots\ldots\ldots\ldots\ldots (9).$$

Since the general equation to the family of surfaces can involve only squares of the coordinates, as is evident from the equations to the generator, we may represent it by

$$f(x, y, z) = 0 \ldots\ldots\ldots\ldots\ldots (10).$$

Let us put

$$\frac{df}{dx} = U, \quad \frac{df}{dy} = V, \quad \frac{df}{dz} = W,$$

$$\frac{d^2f}{dx^2} = u, \quad \frac{d^2f}{dy^2} = v, \quad \frac{d^2f}{dz^2} = w,$$

$$\frac{d^2f}{dy\,dz} = u', \quad \frac{d^2f}{dz\,dx} = v', \quad \frac{d^2f}{dx\,dy} = w',$$

$$\frac{d^3f}{dx^3} = a, \quad \frac{d^3f}{dy^3} = b, \quad \frac{d^3f}{dz^3} = c,$$

$$\frac{d^3f}{dy^2\,dz} = a', \quad \frac{d^3f}{dz^2\,dx} = b', \quad \frac{d^3f}{dx^2\,dy} = c',$$

$$\frac{d^3f}{dy\,dz^2} = a_1, \quad \frac{d^3f}{dz\,dx^2} = b_1, \quad \frac{d^3f}{dx\,dy^2} = c_1,$$

$$\frac{d^3f}{dx\,dy\,dz} = h.$$

Differentiating (10) we have

$$U dx + V dy + W dz = 0,$$

and therefore, since dx, dy, dz, may be supposed to be the projections of an elementary arc of the generator on the coordinate axes, we get, by (9),

$$\lambda U + \mu V + \nu W = 0 \ldots\ldots\ldots (11).$$

Differentiating (11) twice on the same hypothesis, we shall obtain the two following equations:

$$\lambda^2 u + \mu^2 v + \nu^2 w + 2\mu\nu u' + 2\nu\lambda v' + 2\lambda\mu w' = 0 \ldots\ldots\ldots (12),$$

$$\lambda^3 a + \mu^3 b + \nu^3 c + 3\mu^2\nu a' + 3\nu^2\lambda b' + 3\lambda^2\mu c' + 3\mu\nu^2 a_1 + 3\nu\lambda^2 b_1 + 3\lambda\mu^2 c_1 + 6\lambda\mu\nu h = 0 \ldots\ldots\ldots\ldots (13).$$

The elimination of λ, μ, ν, between (11), (12), (13), will give the partial differential equation to the family of the wave-surface.

Eliminating ν between (11) and (12), we shall get

$$\lambda^2 A_\nu + \mu^2 B_\nu + 2\lambda\mu C_\nu = 0 \ldots\ldots\ldots (14),$$

where
$$A_\nu = W^2 u + U^2 w - 2WUv',$$
$$B_\nu = W^2 v + V^2 w - 2WVu',$$
$$C_\nu = UVw - WUu' - WVv' + W^2 w'.$$

Eliminating ν between (11) and (13), we shall get

$$\lambda^3 D_\nu + 3\lambda^2\mu D'_\nu + 3\lambda\mu^2 E'_\nu + \mu^3 E_\nu = 0 \ldots\ldots (15),$$

where
$$D_\nu = W^3 a - U^3 c + 3WU^2 b' - 3W^2 U b_1,$$
$$D'_\nu = -U^2 Vc + WU^2 a_1 + 2UVWb' - VW^2 b_1 + 3W^3 c' - 2W^2 Uh,$$
$$E'_\nu = -V^2 Uc + WV^2 b' + 2UVWa_1 - UW^2 a' + 3W^3 c_1 - 2W^2 Vh,$$
$$E_\nu = W^3 b - V^3 c + 3WV^2 a_1 - 3W^2 Va'.$$

Multiplying (14) by λ, μ, successively, we have

$$\lambda^3 A_\nu + \lambda\mu^2 B_\nu + 2\lambda^2\mu C_\nu = 0 \ldots\ldots\ldots (16),$$
$$\lambda^2\mu A_\nu + \mu^3 B_\nu + 2\lambda\mu^2 C_\nu = 0 \ldots\ldots\ldots (17).$$

Multiplying (15) by $A_\nu B_\nu$, and then eliminating λ^3, μ^3, by means of (16) and (17), we shall get, dividing the resulting

equation by $\lambda\mu$,

$$\lambda(3A_\nu B_\nu D'_\nu - 2B_\nu C_\nu D_\nu - A^2{}_\nu E_\nu) = -\mu(3A_\nu B_\nu E'_\nu - 2A_\nu C_\nu E_\nu - B^2{}_\nu D_\nu) \ldots (18).$$

By symmetry we have also

$$\mu(3A_\lambda B_\lambda D'_\lambda - 2B_\lambda C_\lambda D_\lambda - A^2{}_\lambda E_\lambda) = -\nu(3A_\lambda B_\lambda E'_\lambda - 2A_\lambda C_\lambda E_\lambda - B^2{}_\lambda D_\lambda) \ldots (19),$$

and

$$\nu(3A_\mu B_\mu D'_\mu - 2B_\mu C_\mu D_\mu - A^2{}_\mu E_\mu) = -\lambda(3A_\mu B_\mu E'_\mu - 2A_\mu C_\mu E_\mu - B^2{}_\mu D_\mu) \ldots (20).$$

Multiplying together the equations (18), (19), (20), we get

$$\begin{aligned} 0 = &(3A_\lambda B_\lambda D'_\lambda - 2B_\lambda C_\lambda D_\lambda - A^2{}_\lambda E_\lambda) \\ &(3A_\mu B_\mu D'_\mu - 2B_\mu C_\mu D_\mu - A^2{}_\mu E_\mu) \\ &(3A_\nu B_\nu D'_\nu - 2B_\nu C_\nu D_\nu - A^2{}_\nu E_\nu) \\ +&(3A_\lambda B_\lambda E'_\lambda - 2A_\lambda C_\lambda E_\lambda - B^2{}_\lambda D_\lambda) \\ &(3A_\mu B_\mu E'_\mu - 2A_\mu C_\mu E_\mu - B^2{}_\mu D_\mu) \\ &(3A_\nu B_\nu E'_\nu - 2A_\nu C_\nu E_\nu - B^2{}_\nu D_\nu), \end{aligned}$$

which is the symmetrical partial differential equation to the family of the wave-surface.

It may be observed that

$$\begin{gathered} B_\lambda = A_\nu, \quad B_\mu = A_\lambda, \quad B_\nu = A_\mu, \\ E_\lambda = -D_\nu, \quad E_\mu = -D_\lambda, \quad E_\nu = -D_\mu, \\ WA_\lambda = VC_\lambda + UC_\mu = WB_\mu, \\ UA_\mu = WC_\mu + VC_\nu = UB_\nu, \\ VA_\nu = UC_\nu + WC_\lambda = VB_\lambda. \end{gathered}$$

From the nature of the preceding investigation it is evident that the differential equation to the family of the wave-surface is coincident in form with the differential equation to skew surfaces generated by the motion of a straight line passing through three given directors, the partial differential coefficients in the case of the latter surfaces involving simple powers of x, y, z, as those in the case of the family of the wave-surface involve squares.

Cambridge, Dec. 23, 1851.

INSTANTANEOUS PLANES AND LINES IN A BODY REVOLVING ABOUT A FIXED POINT.

By William Walton.

Conceive a rigid body to be in motion under the action of any forces, one point of the body being fixed. We know that the actual motion of the body at any instant may be *constructed* by impressing upon it simultaneously three motions, each of which, taken singly, would correspond to an angular velocity, common to all its molecules, about one of three fixed coordinate axes passing through the fixed point. These three angular velocities are ordinarily represented by the symbols ω_x, ω_y, ω_z, and may be called the *constructive* angular velocities of the body or of its molecules. Let the *actual* angular velocities of any molecule m, of which the coordinates are x, y, z, be denoted by Ω_x, Ω_y, Ω_z.

(I.) Let us determine the relations subsisting between the constructive and the actual angular velocities of any molecule.

It is proved in the ordinary treatises that

$$\frac{dx}{dt} = z\omega_y - y\omega_z, \quad \frac{dy}{dt} = x\omega_z - z\omega_x, \quad \frac{dz}{dt} = y\omega_x - x\omega_y;$$

hence
$$(y^2 + z^2)\,\Omega_x = y\frac{dz}{dt} - z\frac{dy}{dt}$$
$$= (y^2 + z^2)\,\omega_x - x(y\omega_y + z\omega_z),$$

and therefore

$$\Omega_x = \omega_x - \frac{x}{y^2 + z^2}(y\omega_y + z\omega_z) \ldots\ldots\ldots (1).$$

By similar reasoning, we have also

$$\Omega_y = \omega_y - \frac{y}{z^2 + x^2}(z\omega_z + x\omega_x) \ldots\ldots\ldots (2),$$

$$\Omega_z = \omega_z - \frac{z}{x^2 + y^2}(x\omega_x + y\omega_y) \ldots\ldots\ldots (3).$$

Thus the actual angular velocities of any proposed molecule are expressed in terms of the constructive angular velocities and the coordinates of position. It is obvious from these formulæ that the constructive and actual angular velocities of a molecule of the body are generally different.

(II.) If we put $x = 0$, then, from (1), we see that $\Omega_x = \omega_x$. Similarly, if $y = 0$, $\Omega_y = \omega_y$, and, if $z = 0$, $\Omega_z = \omega_z$. Thus it

appears that the actual angular velocities of all molecules in the fixed coordinate planes, about the axes respectively normal to them, coincide with the constructive angular velocities.

We proceed now to prove the two following theorems:

(1) That, if instantaneous planes be drawn through the axes of x, y, z, perpendicular to the projections of the instantaneous axis upon the planes of yz, zx, xy, respectively, the actual and constructive angular velocities of all the molecules in these instantaneous planes about the axes of x, y, z, respectively, will be equal; and

(2) That there exist three instantaneous lines, equally inclined to the instantaneous axis, such that all molecules in them have equal constructive and actual angular velocities about the axes of y and z, z and x, x and y, respectively.

(III.) If we assume that

$$y\omega_y + z\omega_z = 0,$$

then, from (1), we see that $\Omega_x = \omega_x$.

But the equations to the instantaneous axis are

$$\frac{x}{\omega_x} = \frac{y}{\omega_y} = \frac{z}{\omega_z}.$$

Hence it appears that, if we draw a plane through the axis of x at right angles to the projection of the instantaneous axis on the plane of yz, the actual and constructive angular velocities of all molecules in this plane about the axis of x are coincident. Analogous remarks are applicable to the axes of y and z.

Let the three instantaneous planes thus determined be called the planes X, Y, Z.

(IV.) It is evident, from what has been said above, that all molecules lying in the line of intersection of the planes Y and Z will have equal actual and constructive angular velocities about the axes both of y and of z. The equations to this instantaneous line are evidently

$$\left.\begin{aligned} x\omega_x + y\omega_y &= 0 \\ x\omega_x + z\omega_z &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (4).$$

The equations to the other two analogous instantaneous lines are evidently

$$\left.\begin{aligned} y\omega_y + z\omega_z &= 0 \\ y\omega_y + x\omega_x &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (5),$$

and
$$\left.\begin{aligned} z\omega_z + x\omega_x &= 0 \\ z\omega_z + y\omega_y &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (6).$$

Let these three instantaneous lines be represented by X', Y', Z'.

(V.) Let λ, μ, ν, be the inclinations of X', Y', Z', respectively, to the instantaneous axis. Then it is easily seen that, ω denoting the angular velocity of the body about the instantaneous axis,

$$\omega^2 \cos^2\lambda = \omega^2 \cos^2\mu = \omega^2 \cos^2\nu = \frac{1}{\frac{1}{\omega_x^2} + \frac{1}{\omega_y^2} + \frac{1}{\omega_z^2}},$$

and that accordingly the three instantaneous lines are equally inclined to the instantaneous axis.

(VI.) Let α, β, γ, be the angles between (Y', Z'), (Z', X'), (X', Y'), respectively. Then

$$\cos\alpha = \frac{\frac{1}{\omega_y^2} + \frac{1}{\omega_z^2} - \frac{1}{\omega_x^2}}{\frac{1}{\omega_x^2} + \frac{1}{\omega_y^2} + \frac{1}{\omega_z^2}},$$

$$\cos\beta = \frac{\frac{1}{\omega_z^2} + \frac{1}{\omega_x^2} - \frac{1}{\omega_y^2}}{\frac{1}{\omega_x^2} + \frac{1}{\omega_y^2} + \frac{1}{\omega_z^2}},$$

$$\cos\gamma = \frac{\frac{1}{\omega_x^2} + \frac{1}{\omega_y^2} - \frac{1}{\omega_z^2}}{\frac{1}{\omega_x^2} + \frac{1}{\omega_y^2} + \frac{1}{\omega_z^2}}$$

These expressions shew that no two of the instantaneous lines can ever degenerate into one.

(VII.) It may readily be ascertained from the equations (1), (2), (3), that there is no molecule in the body of which the constructive and actual angular velocities coincide relatively to the three axes taken simultaneously.

Cambridge, Jan. 23, 1852.

ON THE METHOD OF VANISHING GROUPS.

By JAMES COCKLE.

[*Continued from* Vol. VI. p. 178.]

XVIII. Let $\Sigma_m.H^n + A\xi + B$ denote that state of a function which next precedes its final reduction. Then if A be of either of the forms

$$h_1^{n-1} + h_2^{n-1}, \text{ or } h_1^{n-1} + h_2^{n-1} + h_3^{n-1} + h_4^{n-1},$$

to say nothing of others, the requirements of the Method of Vanishing Groups are satisfied. The latter form is the one which we have hitherto supposed to hold, and in the case of $n = 3$ there is no essential difference between them; but for biquadratic and higher functions the former must be exclusively adopted. We shall thereby most materially reduce the number of indeterminate quantities.

XIX. Thus, in III., we may take the reduced value

$$(u_1) = 1 + 2(3.2^2 - 1) = 3.2^3 - 1 = 23,$$

and the other values of u will be almost inconceivably diminished.* The same reduction may be applied to w in IV., and, in short, to all the higher functions.

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* We shall have, for instance,

$$(u_9) = 3.2^{17} - 1,$$

in place of the value given in XIII. When considered in reference to its application to the Theory of Equations, the essentially *indeterminate* character of the method must be borne in mind. The more recent improvements in that theory have all partaken of this character; and, perhaps, where transformation only is in question it is useless to go beyond it. But, whenever we are considering the complete solution of a system of equations, it may be doubtful whether some general *determinate* method, some extension of that of Lagrange or Vandermonde for example, might not be successfully applied to illustrate and explain the principles of the solution. I am inclined to think that a general determinate method of my own (which I term the Method of Symmetric Products, and respecting which some information will be found at pp. 226—229 and 486, 487, of vol. LII. of the *Mechanics' Magazine*, and at the places there cited) possesses advantages in this respect. I have recently completed an investigation of its application to biquadratics. The results will be seen in the following detailed solution of a biquadratic by

The Method of Symmetric Products.

Let y_1, y_2, y_3, and y_4 be the roots of

$$y^4 + Ay^3 + By^2 + Cy + D = 0.$$

It is required to find three unequal values of the expression

$$Y_r = y_1 + \alpha_r y_2 + \beta_r y_3 + \gamma_r y_4,$$

whose product shall be a symmetric function of $y_1, \ldots, y_4$.

XX. Many of our foregoing results may be arrived at *synthetically*. If we denote by n_n the number of indeter-

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Let Y_1, Y_2, and Y_3 be the values. Then the terms of the form y_1^3 give the following conditions of symmetry:

$$1 = a_1a_2a_3 = \beta_1\beta_2\beta_3 = \gamma_1\gamma_2\gamma_3;$$

those of the form $y_1^2y_2$ give

$$\Sigma.a = \Sigma.\beta = \Sigma.\gamma = \Sigma.a_1a_2 = \Sigma.\beta_1\beta_2 = \Sigma.\gamma_1\gamma_2$$
$$= \Sigma.a_1a_2\beta_3 = \Sigma.a_1a_2\gamma_3 = \Sigma.a_1\beta_2\beta_3$$
$$= \Sigma.a_1\gamma_2\gamma_3 = \Sigma.\beta_1\beta_2\gamma_3 = \Sigma.\beta_1\gamma_2\gamma_3;$$

and those of the form $y_1y_2y_3$ give

$$\Sigma.a_1\beta_2 + a_2\beta_1 = \Sigma.a_1\gamma_2 + a_2\gamma_1$$
$$= \Sigma.\beta_1\gamma_2 + \beta_2\gamma_1 = \Sigma.(a_1\beta_2 + a_2\beta_1).\gamma_3.$$

Let E be the coefficient of $y_1^2y_2$. Then the above relations shew that (a_1, a_2, and a_3), (β_1, β_2, and β_3) and (γ_1, γ_2, and γ_3) are systems of roots of the equation

$$z^3 - Ez^2 + Ez - 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (a).$$

From the form of the roots of (a) we are at liberty to assume

$$a_1 = m^{-1}, \quad a_2 = m, \quad \text{and} \quad a_3 = 1.$$

Substitute these values in the equations

$$\Sigma.\beta - \Sigma.a_1a_2\beta_3 = 0 = \Sigma.\gamma - \Sigma.a_1a_2\gamma_3,$$

and we obtain

$$\beta_2 = m\beta_1, \quad \gamma_2 = m\gamma_1,$$

and hence, by means of the first conditions of symmetry,

$$\beta_3 = m^{-1}\beta_1^{-2}, \quad \gamma_3 = m^{-1}\gamma_1^{-2}.$$

By substitution in the expressions

$$\Sigma.\beta_1\beta_2\gamma_3 \text{ and } \Sigma.\beta_1\gamma_2\gamma_3,$$

we now obtain

$$\left(\frac{\beta_1}{\gamma_1}\right)^2 + 2\frac{\gamma_1}{\beta_1} = \left(\frac{\gamma_1}{\beta_1}\right)^2 + 2\frac{\beta_1}{\gamma_1} = E,$$

and consequently

$$\beta_1\gamma_1^{-1} = \pm 1, \text{ and } E = 3 \text{ or } -1, \text{ and } \beta_3 = \gamma_3.$$

The first value of E is useless, since it would render the values of Y equal. Hence, adopting the last, we see that (a) is equivalent to

$$(z+1)^2(z-1) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (b);$$

and that

$$m = -1 = a_2 = a_1 = \beta_3 = \gamma_3;$$

and rejecting, as we must do, the positive sign in the expression for $\beta_1\gamma^{-1}$, we have, finally,

$$\beta_1 = -\gamma_1 = -\beta_2 = \gamma_2 = \pm 1 \text{ indifferently.}$$

It will perhaps be better to make $\beta_1 = 1$. All the conditions of symmetry are now satisfied, and the symmetric product is

$$Y_1Y_2Y_3 = -8C + 4AB - A^3 = P_3.$$

Let x be the root of any biquadratic; and, assuming

$$y = \lambda x + x^2,$$

transform, by the known processes, the biquadratic in x into another in y, in which λ is determined (by a cubic, the use and meaning of the several

minates necessary for the reduction of a function to the form

$$h_1^n + h_2^n + \ldots + h_x^n,$$

we have (besides the anomalous result $1_x = y$, where y is indeterminate) the following:

$$2_{x+1} = 2_x + 1,\ 2_x = x;\quad 3_{x+1} = 2.3_x + 1,\ 3_x = 3.2^x - 1;$$

$$4_{x+1} = 3.2^{1+2.4_x} - 1;\quad 5_{x+1} = 3.2^{1+2.5_x} - 1;\ \&c.$$

The above two integrals are corrected. In forming the remaining equations we must employ the 'reduced' values of 4_1, 5_1, &c. By actual development we see that

$$3_x = 1 + 2\{1 + 2[1 + 2(1 + \ldots . 2(1 + 2^2)\ldots)]\},$$

where there are x brackets; and, consequently,

$$3_x = 1 + 2 + 2^2 + \ldots + 2^{x-1} + 2^{x+1} = 3.2^x - 1,$$

which confirms our preceding result.

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roots of which I shall not now stop to discuss), so as to render $P_3 = 0$. Let a be any constant quantity, then, from the structure of P_3, we see that that product, and each of its factors, remains unchanged in value when $y + a$ or y' is written in place of y. Let $4a = -A$, and denote by an accent the changes corresponding to the change of y into y'; then we have

$$P_3' = 0,\ A' = 0,\ \text{and } \textit{consequently } C' = 0.$$

Hence if, by means of $Y_1' = 0$ and $A' = 0$, we eliminate y_1' and y_3' from $C' = 0$, we have a result which admits of a (cubic) solution of the form $y_3 = ry_1$; and y_1 will be determined by substitution in

$$\Sigma.y_1'y_2' = B',$$

and by means of a quadratic. It is of course immaterial in what order the roots are eliminated; but y' and y and x are completely determined—the latter by known processes.

The function P_3 is a critical function; it possesses this property: that if y and x be connected by the relation

$$y = a + f(x),$$

where f is integral and rational, then P_3 is free from a whatever be the value of a or the form of f. The corresponding symmetric product for cubics (viz. $P_3 = A^3 - 3B$) is also critical. These are facts which must not escape us when we come to consider the further application of the Method of Symmetric Products. Assuming the impossibility of solving equations of the fifth and higher degrees, a theory of *conjugate* equations extending to all degrees would seem to be fairly within the grasp of Algebra.

If, in the above investigations, we take the values of a as given above, we find that

$$\Sigma.a_1\beta_2 + a_2\beta_1 = \Sigma.a_1\gamma_3 + \gamma_1 a_3,\ \text{gives } \beta_3 = \gamma_3;$$

$$\Sigma.\beta_1\gamma_2 + \beta_2\gamma_1 = \Sigma.(a_1\beta_2 + a_2\beta_1)\,\gamma_3,\ \text{gives } \gamma_3 = -1;$$

and $$\Sigma.\beta_1\gamma_2\gamma_3 = -1,\ \text{gives } \beta_1\gamma_3 + \beta_3\gamma_1 = -2.$$

This confirms the above results. The last relation shews how the values of β and γ are connected.

XXI. In its application to the solution of a simultaneous quadratic and n^{ic} equation, the process of the method is identical in principle with that indicated by Mr. Sylvester in a note at a previous portion of this *Journal* (vol. VI. pp. 14–15); and it gives that which is substantially equivalent to what Mr. Sylvester terms a 'linear solution' of the quadratic.

It must never be forgotten that we may always introduce as many indeterminates as we please into any equation or system of equations. When we have only one equation to deal with, the roots of the transformed equation are in general linear functions of the indeterminates, but non-linear with respect to the roots of the original equation. The theory of systems of *determinate* equations is yet in its infancy, but it is certain that there, as in the theory of single equations, the limitation of the number of solutions must extend implicitly to the transformed equations in their indeterminate form, and, by causing the occurrence of vanishing fractions, give rise to illusory results. I shall not, however, at the present moment pursue the subject further than to suggest it as a topic for discussion.

XXII. Let U and U' be homogeneous quadratic functions. Then, in general,

$$U = \gamma(U) = h_1^2 + h_2^2 + u,$$
$$U' = \gamma(U') = h'^2_1 + h'^2_2 + u',$$

and, if the coefficient of ξ' in U and U' be unity, we have

$$h'_1 = h_1 + bh_2 + c, \text{ and } h'_2 = dh_2 + e;$$

consequently, if we form the expression

$$(\lambda - 1)\, U + U' = \lambda V,$$

we have a result that may be represented by

$$\lambda h_1^2 + 2bh_1h_2 + (\lambda - 1 + a)\, h_2^2 + 2ch_1 + 2fh_2 + l.$$

XXIII. Let λ_1 and λ_2 be the roots of the equation

$$\lambda(\lambda - 1 - a) = b^2,$$

and let $\quad b\lambda_1^{-1} = a_1 + a_2, \text{ and } b\lambda_2^{-1} = a_1 - a_2;$

then we have two values of V included in an equation of the form

$$V = \{h_1 + (a_1 \pm a_2)\, h_2\}^2 + 2Ch_1 + 2Fh_2 + p.$$

XXIV. Suppose that

$$h_1 + a_1h_2 = v, \quad a_2h_2 = w, \quad 2(F - Ca_1) = 2F'a_2,$$
$$C + F' = 2\lambda^{-1}m, \text{ and } C - F' = 2\lambda^{-1}n,$$

then we obtain

$$V = (v \pm w)^2 + 2\lambda^{-1}m(v + w) + 2\lambda^{-1}n(v - w) + p.$$

XXV. We now see that the solution of the system $U = 0 = U'$ is reduced to that of

$$(v + w + \lambda_1^{-1}m)^2 + 2\lambda_1^{-1}n(v - w) + p - \lambda_1^{-2}m^2 = 0,$$

$$(v - w + \lambda_2^{-1}n)^2 + 2\lambda_2^{-1}m(v + w) + q - \lambda_2^{-2}n^2 = 0.$$

In the particular case where U and U' involve only *two* unknowns this reduction enables us to effect the solution without having to solve an equation higher than a *cubic*. Its bearing upon the general case remains for consideration.

[*To be continued.*]

Erratum.—Vol. VI. p. 178, line 4, *for h read* h.

2, *Pump Court, Temple,*
Sept. 30, 1851.

ON SOME GEOMETRICAL CONSTRUCTIONS.

By H. J. S. Smith, Fellow of Balliol College, Oxford.

If a geometrical curve be completely traced on a sheet of paper, the principles of the Theory of Transversals enable us to assign its tangent line and radius of curvature at any point, without supposing its equation known, and without employing any operations excluded from the sphere of elementary geometry.

The construction given by M. Chasles for this purpose is the following. Let m be the point on the curve, M any point assumed in the plane, mp, mq, two transversals, and MP, MQ, two parallels to them. Let also p, q, P, Q, denote the continued products of the segments on mp, mq, MP, MQ, respectively, excepting the evanescent segments on mp, mq. Then if we take on mp, mq, two lines respectively proportional to $\frac{P}{p}, \frac{Q}{q}$, the line joining their extremities shall be parallel to the tangent at m. To find the osculating circle, let t denote the continued product of the segments on the tangent at m, excepting the two evanescent segments, T the continued product of the segments on MT drawn parallel to mt; then, if, on any transversal

mp, we take mc equal to $\frac{T}{t} \cdot \frac{p}{P}$, the point c shall lie on the osculating circle. For the diameter of this circle we have the expression $\frac{T}{t} \cdot \frac{n}{N}$; n, N denoting products of segments on the normal and on a parallel to it.

If the point be a double point, the preceding constructions fail; but by slightly modifying them, we may determine the two tangents and two radii of curvature, if the point be nodal: or, if it be conjugate, we can assign the elements of an ellipse, whose imaginary asymptotes shall be the imaginary tangents in question; that is to say, an ellipse concentric, similar, and similarly placed, with the evanescent conic formed by the conjugate point. In this case the two radii of curvature are in general imaginary and therefore cannot be constructed: but, since they are conjugate imaginary magnitudes, any rational symmetrical functions of the two (for example, the rectangle under them, or their harmonic or arithmetic mean), may readily be determined. The process to be employed is as follows. Take three transversals mp, mq, mr, and three parallels to them MP, MQ, MR, and let MR cut mp, mq in A, B, and the two tangents in θ_1, θ_2. We shall have

$$A\theta_1 . A\theta_2 = (mA)^2 . \frac{R}{r} . \frac{p}{P}, \quad B\theta_1 . B\theta_2 = (mB)^2 . \frac{R}{r} . \frac{q}{Q}.$$

Now if the point m be conjugate, the products $A\theta_1$, $A\theta_2$, $B\theta_1$, $B\theta_2$, are essentially positive; and if it be nodal we can ensure their being so, by taking the three transversals mp, mq, mr, in one and the same pair of vertically opposite angles. Hence, if we put

$$(mA)^2 \frac{R}{r} . \frac{p}{P} = a^2, \quad (mB)^2 \frac{R}{r} . \frac{q}{Q} = b^2,$$

the lines a and b can always be constructed. Therefore to determine θ_1, θ_2, describe two circles round A, and B, with radii a and b, respectively. The radical axis of these circles will intersect MR at o the middle point of θ_1, θ_2; and any circle of the system orthogonal to the two circles (A) and (B), (*i.e.* any circle having its centre on the radical axis and its radius equal to the tangential distance of its centre from either of those two circles), will intersect MR in θ_1, θ_2. If (A) and (B) intersect in real points, their radical axis is instantly found; but in this case θ_1 and θ_2 are always

imaginary. Let s_1, s_2 be the points in which the radical axis is cut by any one of the orthogonal circles; on mr take mo' a mean proportional between os_1 and os_2 (*i.e.* equal to the tangential distance of o from any one of the orthogonal circles): the ellipse having its centre at m, and mo, mo' for semi-conjugate diameters, will have the two imaginary tangents for its asymptotes. If (A) and (B) touch, the points $\theta_1\theta_2$ coincide in o, and the double point becomes a cusp, having mo for its tangent. Lastly, if (A) and (B) intersect in imaginary points, the radical axis, though not immediately given, can always be determined by the ruler alone, and in this case, $\theta_1\theta_2$ being always real, the tangents $m\theta_1$, $m\theta_2$ can be directly constructed.

The direction of the tangents once ascertained, the radii of curvature may be immediately found. In fact, if we denote by c_1, c_2 the chords intercepted on mp by the two circles, and by $\theta_1\theta_2$ the two points in which the tangents are cut by MP parallel to mp, we have

$$c_1 = \frac{\theta_1\theta_2}{m\theta_1} \cdot \frac{T_1}{t_1} \cdot \frac{p}{P}, \quad c_2 = \frac{\theta_1\theta_2}{m\theta_2} \cdot \frac{T_2}{t_2} \cdot \frac{p}{P};$$

and by making mp coincide successively with the two normals, we get the values of the two diameters of curvature. Or we may first determine one, and then obtain the second by the proportion, which is easily demonstrated,

$$R_1 : R_2 :: \frac{T_1}{t_1} : \frac{T_2}{t_2}.$$

If the point be triple the determination of the directions of its tangents, which in analysis depends on the solution of a cubic equation, is not in general possible by the intersections of right lines and circles. The problem in its simplest form is this: Given three points on a right line ABC, and given the products $A\theta_1 . A\theta_2 . A\theta_3$, $B\theta_1 . B\theta_2 . B\theta_3$, $C\theta_1 . C\theta_2 . C\theta_3$, find $\theta_1\theta_2\theta_3$. But whatever the order of the point, if the direction of its tangents be once known, the construction of its radii of curvature is very easy. If, for example, the order of the point be r, the chord determined on mp, by the circle tangent to $m\theta_1$, is readily seen to be given by the equation

$$c_1 = \frac{T_1}{t_1} \frac{p}{P} \frac{\theta_1\theta_2 . \theta_1\theta_3 \ldots \theta_1\theta_r}{(m\theta_1)^{r-1}};$$

which chord is therefore imaginary for an imaginary tangent, as it ought to be.

Returning to the case of double points, we see from the formula

$$c_1 = \frac{\theta_1\theta_2}{m\theta_1} \cdot \frac{T_1}{t_1} \frac{p}{P},$$

that if the two tangents coincide, the osculating circles become simultaneously evanescent, except a fourth segment on the tangent become evanescent also, that is, except the tangent cut the curve in four coincident points at m. In this case the point m is not cuspidal, but is a point of osculation, and possesses two radii of curvature, for which we proceed to give a graphical construction. If D_1D_2 be the two diameters, we find readily enough $D_1D_2 = \frac{T}{t} \cdot \frac{n}{N}$; but the theorem of Newton's, which has hitherto guided us, is perhaps insufficient immediately to furnish a second relation. Such a relation, however, may be obtained by the following considerations. It is well known that the polar conic of a point of inflexion breaks up into two lines: one of these is the tangent at the point of inflexion, the other will be found to be the locus of the harmonic centres of the $n-1$ points in which the curve is cut by a transversal through the point. Exactly in the same way the polar curve of the third order at a point m of the nature here considered, resolves itself into the tangent line and into a conic section; this conic touches the tangent at m, and is the locus of harmonic centres of the $n-2$ points in which the curve is cut by a transversal through m; consequently its curvature at m, multiplied by $n-2$, is precisely the sum of the curvatures sought. Now the diameter of curvature in a conic is to the chord intercepted on the normal, as the rectangle under the segments of a parallel to the tangent is to the rectangle under the segments of the normal chord. Hence, if mp_1 be any radius vector of the conic, p_1p_2 a chord parallel to the tangent at m, the radius of curvature is known as soon as the point p_2 has been constructed. To effect this, take any circle tangent to the conic at m; this circle and the conic being homological, their axis of homology may be first found, and then the line homologous to p_1p_2; this will give the point homologous to p_2, and therefore p_2 itself; in fact, the circle once described, p_2 may be found by the ruler alone. We now know the rectangle under the two radii of curvature, and the harmonic mean between them: the radii may therefore themselves be found by a simple and well-known construction.

It may be observed that the theory of polar curves leads to a construction for the tangent of a curve line, which is different from M. Chasles', and in fact linear. Through the given point P draw four transversals; each of these will cut the curve in $n-1$ points. Take the harmonic centre of each of these four groups with respect to P, and consider the four points thus obtained as determining a conic section passing through P. Pascal's theorem will then determine the tangent to this conic at P; that is, the tangent required. It is unnecessary to give the reciprocal construction, which enables us, when a curve of the n^{th} class is given tangentially, to determine with the ruler alone the point of contact on any one of its tangents, supposed not to be a double tangent. It should however be added, that a method for the linear solution of these two problems has been long since given in a different and less explicit form by M. Poncelet, in his excellent memoirs on the Analysis of Transversals.

The radius of curvature of any point is, of course, by its nature, incapable of linear construction; but if we imagine ourselves to have constructed the normal at any point, and to have determined on it the centre of curvature of the given curve or of any one of its superior or inferior polar curves; and if, in addition, a line parallel to the normal be given, in order that the point at infinity on the normal may be known; we can find linearly the centre of curvature of every single curve of the polar system continued as far upwards as we please. This is a consequence of the following theorem: The distances of the centres of curvature of the successive polar curves from their common tangent form a harmonic progression, commencing at infinity and having the given point for its point of evanescence.

All the preceding methods admit of an easy application to the theory of surfaces. For example, to determine the tangent plane at m, we must draw three transversals mp, mq, mr, not in one plane, and then proceed as in the case of plane curves. If the surface be of the n^{th} order, its tangent plane will determine on it a curve of the same order; the point m being a double point in that curve nodal, cuspidal, or conjugate, according as the contact is hyperbolic, parabolic, or elliptic. Taking the last case, we must determine two conjugate semidiameters of an ellipse concentric, similar, and similarly placed with the evanescent conic in the tangent plane, and therefore with the indicatrix of the point m: the directions and magnitudes of the semi-

axes of this ellipse may now be deduced by a construction of extreme simplicity (vide. Note xxv. on M. Chasles' *History of Geometry*), and therefore the ratio of the principal curvatures, and the traces of the principal normal sections on the tangent plane are known. If now we construct the radius of curvature in either of these normal sections, the square of one semi-axis of the indicatrix is found, and therefore that curve may be regarded as completely determined. It hence appears, that to find the tangent plane and the indicatrix of any point, it is requisite to draw sixteen transversals; not that so many are absolutely essential, but the trouble is rather increased than lessened by taking fewer.

From their connexion with the present subject the following geometrical demonstrations of Meunier's and Euler's theorems on curvature may find a place here. If we take a point P on a curve line, and if we consider an evanescent chord drawn parallel to the tangent at P as an infinitesimal of the first order, this chord will be bisected by the normal at P; that is to say, it will be divided into two segments whose difference will be infinitesimal of the second order. Moreover, if we take any sagitta perpendicular to the chord, and intersecting it in a point distant only by an infinitesimal of the second order from its centre, it is readily seen that the square of either segment of the chord, divided by the sagitta, may be taken to represent the diameter of curvature of the evanescent arc. Hence, if we take two plane sections of a surface intersecting in an evanescent chord, the radii of curvature of the evanescent arcs are inversely as any two sagittæ perpendicular to the chord, and bisecting it approximately. If, now, one of the sections be a normal one, we may take for the sagitta in that section the intercept on the normal to the surface. Consequently the triangle formed by joining the extremities of the two sagittæ will be right-angled; and therefore the radius of the oblique section is equal to the orthogonal projection of the normal radius upon the plane of the oblique section, which is Meunier's theorem.

It follows also from what has been said, that if we draw a plane parallel to a tangent plane, and distant from it by an infinitesimal of the second order, the curve surface will, in general, determine upon this plane an evanescent hyperbolic or elliptic oval; and that the chords of the oval, being themselves infinitesimals of the first order, will be bisected within an infinitesimal of the second order at the point at which the normal meets the plane, and which we will call C.

We may therefore consider the oval as a central curve having its centre at C: it only remains to shew that it is a conic section. This may be done as follows: Every transversal passing through C and lying in the plane of the oval, will cut the surface in two points belonging to the oval, and in $n-2$ points whose distance from C is infinitely great compared with that of the two first points. Now, if for a moment we consider the diameters of the oval to be finite, the remaining $n-2$ points will lie at infinity, and therefore an infinitely magnified representation of the section we are considering would consist of a finite central conic, replacing the oval, and of the line at infinity $n-2$ times repeated, replacing the $n-2$ branches which lie at a finite distance from C. Since, then, the radii of curvature of the normal sections vary as the squares of the diameters of the evanescent oval, they vary as the squares of the central radii vectores of a conic section.

If there be a double line upon the surface we can construct the two tangent planes at any point m by taking two plane sections passing through m and constructing the tangents of the double points. Each of these tangent planes will cut the surface in a curve having a triple point at m; but as the direction of one of the three tangents is known *a priori*, being the intersection of the tangent planes, the directions of the remaining two may be found by the construction used for double points; consequently the directions of the tangents to the principal sections on each sheet of the surface are known, and the principal radii of curvature may be determined by the construction before given for finding either radius of curvature at a double point. The two indicatrices at the point m may therefore be considered as ascertained in magnitude and position.

If the osculating plane and radius of curvature of the double line itself be required, they may be obtained very simply by a method to be given below.

If the singular line be of the r^{th} order ($r > 2$), a little consideration will shew that though we cannot determine the tangent planes by any elementary construction, yet, if we assume these planes as known, the indicatrices upon each sheet may be found as easily as if the point were not singular. This is the more remarkable, since the expressions given by analysis for the principal radii of curvature at such points appear to be of great complexity.

Let us now take a point m on a curve surface where two sheets of the surface meet and have a common tan-

gent plane. This tangent plane will intersect the surface in a curve having a quadruple point at m; but the directions of the four tangents may always be ascertained by a quadratic construction. For at such a point the polar surface of the third order will resolve itself into the tangent plane and into a surface of the second order. And it may be shewn that the two generatrices (real or imaginary) of that surface which lie in the tangent plane are in involution with the two pair of asymptotes of the two indicatrices; that is, with the four tangents before mentioned. Now these two generatrices may be determined by means of the theory of homological figures, since that theory enables us to assign a pair of semi-conjugate diameters of a section of the surface of the second order parallel to the tangent plane at m, whence the directions of the asymptotes of that section become known, and therefore the two generatrices required. The problem now will be: Given four points in a line $PQRS$, and the four products $P\theta_1.P\theta_2.P\theta_3.P\theta_4$, &c., determine $\theta_1\theta_2\theta_3\theta_4$, a pair of points G_1G_2 being also given which form an involution with the two pairs $\theta_1\theta_2$ and $\theta_3\theta_4$. It is plain that, A being any point whatever, any symmetrical function of the distances $A\theta_1$, $A\theta_2$, $A\theta_3$, $A\theta_4$, may be constructed. Hence H_1H_2, the harmonic centres of the four points $\theta_1\theta_2\theta_3\theta_4$ with respect to G_1G_2, are known. But H_1H_2 form a pair of points in involution with the two pair sought: and therefore H_1H_2 together with G_1G_2 completely determine the involution. Therefore the centre and foci of the system are known, and consequently $\theta_1\theta_2$ and $\theta_3\theta_4$ may be now quadratically determined. Points of the nature here considered may exist isolated on a curve surface; but if there be a continuous series of them, we shall have a line along which two sheets of the surface envelope one another (not a cuspidal line, for any transversal plane will determine a section having not cusps, but points of osculation at its intersections with the singular line), and at any point on such a line the two generatrices before mentioned will be found to coincide: and consequently the surface of the second order will degenerate into a cone. The side of this cone, existing in the tangent plane at m, may be determined by proceeding as in the general case: and then, instead of a pair of points in involution with $\theta_1\theta_2\theta_3\theta_4$, we shall have one focus of that involution given. The second focus may next be constructed (being the harmonic centre of $\theta_1\theta_2\theta_3\theta_4$ with respect to the given focus), and the problem becomes quadratic as before.

Lastly, let there be a point on a curve surface having a tangent cone of the second order. It will be possible to determine three conjugate diameters of that cone. For, take any two planes passing through the vertex, and having constructed the tangent lines of the double points in those planes, take the harmonic conjugates of the line of intersection of the two planes with respect to each pair of tangents. This will give the plane conjugate to the line of intersection; and by taking any two lines harmonically conjugate with respect to the two tangent lines existing in that plane, we shall obtain the directions of the three semidiameters required. Likewise, their ratios, or rather the ratios of their squares, may be found, since the two sides of the cone in each conjugate plane may be constructed. Hence, we may deduce the directions and the ratios of the squares of the principal semiaxes of the cone. But this determination, involving the solution of a cubic equation, requires the construction of a conic, and is consequently not within the limits of elementary geometry. (Vide the Note on M. Chasles' *History*, already quoted, and a paper by Mr. Townsend in this *Journal*.)

If a curve of double curvature be given in space the principles of the theory of transversals are not immediately applicable: but if we regard it as the intersection of two geometrical surfaces completely given, we may immediately find its tangent, osculating plane, and radius of curvature. The tangent at m is of course determined by the intersection of the two tangent planes; and if we take the two normal sections containing that tangent, and, having constructed their radii of curvature, let fall a perpendicular from m on the line joining the two centres, this perpendicular will represent in magnitude and direction the radius of curvature of the given curve. This (it will be seen) follows at once from Meunier's theorem, or from that known as Hachette's.

LAPLACE'S EQUATION AND ITS ANALOGUES.

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It has been long recognized by mathematicians that the arbitrary portions of the solutions of partial differential equations can, in general, be expressed under three distinct forms. *The first* gives the arbitrary portion of the general solution

as the sum of an infinite number of particular expressions, and, though open to objection, is recommended by the circumstances that it is unaffected by any signs of integration, and is wholly free from arbitrary functions. *The second*, which is due to Laplace, expresses the arbitrary portion by means of Definite Integrals, under the signs of which arbitrary functions occur; and while it is, in the existing state of science, generally unattainable, yet when once arrived at, it has the advantage of enabling us to determine the arbitrary functions with considerable facility. *The third and most common* form is that which exhibits the same arbitrary portion in terms of arbitrary functions without any signs of integration; and, in contrast with the last, while in most cases it *can* be obtained by the aid of the Calculus of Operations, it is objectionable from the difficulty of determining the arbitrary functions. It may be observed that this last form is unique, whereas in each of the two preceding cases we have obviously as many general solutions as we can obtain particular solutions of distinct types. The order in which these forms have been stated seems to be that of their chronological employment, although the reverse of this order is that of their logical filiation.

It is proposed in the following paper to exhibit general solutions of the class of partial differential equations to which the Laplacian equation belongs, under these three several forms. It is obvious that, as this class of equations contains no term in its right-hand member involving only the independent variables, the general solutions will contain no determinate expression, but will reduce themselves to the arbitrary portions solely. It will be found that the particular solutions employed in the first and second forms are duplicate, each indicating a second in close correlation with itself. In the case of the equation of the simplest type, the duplicate solutions are omitted as being evident.

The instruments employed in arriving at these solutions are known by the names of Calculus of Duplets, Triplets, Quaternions, &c., which have of late occupied so considerable a share of the attention of the mathematical world. Frequent reference is made to a paper on Quaternions by Sir William Rowan Hamilton, published in the second volume of the *Proceedings of the Royal Irish Academy*, and the laws of the system of Triplets here employed are analogous to those which govern Quaternions. The writer is not aware that any use has as yet been made of these instruments in connexion with the subject of the integration

of partial differential equations, and believes the forms of the solutions themselves, as found by them, to be new. It will be seen that he has commenced with the equation of the simplest character, the solution of which is familiar to the reader, and by successively engaging the equations as they rise in order, has endeavoured to shew the identity of the method applied to all.

To the end of the paper are appended some additional examples of the applicability of imaginary symbols of operation to integration, and it will be evident from them that such symbols can be used to great advantage when the equations to be integrated are symmetrical.

1. It is known that a general solution of the equation

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} = 0 \quad \ldots\ldots\ldots\ldots \text{(I.)}$$

is given by $U = \Sigma A e^{m_1 x + m_2 y}$,

where m_1, m_2 are connected by the relation

$$m_1^2 + m_2^2 = 0.$$

Now, by the ordinary Calculus of Imaginaries, or *Duplets*, this relation is equivalent to

$$(m_2 + im_1)(m_2 - im_1) = 0,$$

where i is such an operation that

$$i^2 = -1.$$

Thus m_2 has either of values $\pm im_1$, and upon the introduction of these, the value of U becomes

$$\Sigma A e^{m_1 x + im_1 y} + \Sigma B e^{m_1 x - im_1 y},$$

or
$$U = \left\{ \begin{matrix} \Sigma A e^{m_1 x} (\cos m_1 y + i \,.\, \sin m_1 y) \\ + \\ \Sigma B e^{m_1 x} (\cos m_1 y - i \,.\, \sin m_1 y) \end{matrix} \right\} \ldots\ldots (a_1),$$

which is a general solution of (I.) in the *first* form. It is evident that this solution may be condensed, but we shall retain it in its present shape for the sake of symmetry.

Now since the arbitrary constants A, B, and m_1 are independent of each other, we may obviously substitute for A and B *arbitrary functions* of m_1, and the solution just found may then be put under the shape

$$U = \left\{ \begin{matrix} \int \Phi(m_1)\, e^{m_1 x} \,.\, \cos m_1 y \,.\, dm_1 \\ + \\ \int \Psi(m_1)\, e^{m_1 x} \,.\, \sin m_1 y \,.\, dm_1 \end{matrix} \right\} \ldots\ldots (a_1)',$$

the limits of the integrals being supposed independent of x and y. At first sight this might be regarded as a solution in the *second* form, that namely of a definite integral. It seems, however, more just to consider it as merely *another shape* of the first form, since the limits are indeterminate.

The general solution in the *third form* is given by

$$U = \left(\frac{d}{dy} - i\frac{d}{dx}\right)^{-1} \cdot \left(\frac{d}{dy} + i\frac{d}{dx}\right)^{-1} \cdot 0,$$

or is

$$U = \phi(x+iy) + \psi(x-iy) \ldots\ldots\ldots (c_1),$$

and the coincidence of the previous with this is evident.

2. Similarly, a general solution of the higher equation, rendered famous by its connexion with the name of Laplace,

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0 \ldots\ldots\ldots \text{(II.)}$$

is given by

$$V = \Sigma A e^{m_1x + m_2y + m_3z},$$

where m_1, m_2, m_3 are connected by the corresponding relation

$$m_1^2 + m_2^2 + m_3^2 = 0.$$

Now, by the Calculus of *Triplets*, this relation is equivalent to

$$(m_3 + im_1 + jm_2)(m_3 - im_1 - jm_2) = 0,$$

where i and j are such operations that

$$i^2 = -1, \quad j^2 = -1, \quad ij = -ji.$$

Thus m_3 has either of values $\pm(im_1 + jm_2)$, and the solution assumes the form

$$V = \begin{cases} \Sigma A e^{m_1x + m_2y + (im_1 + jm_2)z} \\ + \\ \Sigma B e^{m_1x + m_2y - (im_1 + jm_2)z}. \end{cases}$$

To obtain the form corresponding to (a_1), we assume

$$m_1 = r\cos a, \quad m_2 = r\sin a,$$

and the value of V becomes

$$\Sigma A e^{m_1x + m_2y + i_r \cdot rz} + \Sigma B e^{m_1x + m_2y - i_r \cdot rz},$$

where

$$\left.\begin{aligned} i_r &= i\cos a + j\sin a \\ r^2 &= m_1^2 + m_2^2 \end{aligned}\right\};$$

and a general solution of (II.) in the *first* form is

$$V = \left\{\begin{array}{l} \Sigma A e^{m_1x + m_2y}\left\{\cos\sqrt{(m_1^2+m_2^2)}z + i_r \cdot \sin\sqrt{(m_1^2+m_2^2)}z\right\} \\ + \\ \Sigma B e^{m_1x + m_2y}\left\{\cos\sqrt{(m_1^2+m_2^2)}z - i_r \cdot \sin\sqrt{(m_1^2+m_2^2)}z\right\} \end{array}\right\} \ldots (a_2),$$

the duplicate of which, namely,

$$V = \left\{ \begin{array}{c} \Sigma A e^{\sqrt{(m_1^2+m_2^2)}z}\{\cos(m_1x+m_2y)+i_r.\sin(m_1x+m_2y)\} \\ + \\ \Sigma B e^{\sqrt{(m_1^2+m_2^2)}z}\{\cos(m_1x+m_2y)-i_r.\sin(m_1x+m_2y)\} \end{array} \right\} \ldots(a_2'),$$

is had by an obvious modification.

It may be well to preserve the solutions in these shapes, as it is probable that the quantity i_r and those corresponding to it bear some relation to the character of the problems whose law is expressed by the equation of which this is the solution.

As before, substituting for A and B arbitrary functions of m_1 and m_2, we can throw these solutions into the following shapes :

$$V = \left\{ \begin{array}{c} \iint \Phi(m_1, m_2)\, e^{m_1x+m_2y}.\cos\sqrt{(m_1^2+m_2^2)}\, z\, .\, dm_1 dm_2 \\ + \\ \iint \Psi(m_1, m_2)\, e^{m_1x+m_2y}.\sin\sqrt{(m_1^2+m_2^2)}\, z\, .\, dm_1 dm_2 \end{array} \right\} \ldots(a_2)',$$

with its duplicate

$$V = \left\{ \begin{array}{c} \iint \Phi(m_1, m_2)\cos(m_1x+m_2y)\, e^{\sqrt{(m_1^2+m_2^2)}\, z}.\, dm_1 dm_2 \\ + \\ \iint \Psi(m_1, m_2)\sin(m_1x+m_2y)\, e^{\sqrt{(m_1^2+m_2^2)}\, z}.\, dm_1 dm_2 \end{array} \right\} \ldots(a_2)'',$$

the limits of the integrals in both cases being supposed independent of the quantities x, y, and z, and their order being equal to the number of the quantities m_1, m_2. In this latter respect it will be found that these solutions are only particular cases of a general law.

The same remark applies to the solutions of (II.) just found as to the solution of (I.) represented by the formula $(a_1)'$. They are not to be regarded as solutions of (II.) in the *second form*, but only as *other shapes* of the solutions in the *first form*, since the limits of the integrals are indeterminate.

Poisson has furnished an integral of the Laplacian equation strictly in the *second* form,* namely

$$4\pi V = \left\{ \begin{array}{l} \int_0^{\pi}\int_0^{2\pi} x\Phi(y + x\sin u\cos v\sqrt{-1}, \\ \qquad\qquad z + x\sin u\sin v\sqrt{-1})\sin u\, du dv \\ + \\ \dfrac{d}{dx}\displaystyle\int_0^{\pi}\int_0^{2\pi} x\Psi(y + x\sin u\cos v\sqrt{-1}, \\ \qquad\qquad z + x\sin u\sin v\sqrt{-1})\sin u\, du\, dv \end{array} \right\} \ldots(b_2);$$

* *Mémoires de l'Institut*, 1818.

but he considers its value lessened by the circumstance of its containing imaginaries under the signs of the arbitrary functions, more especially since, in the application of these solutions, the arbitrary functions ought to be discontinuous. The integral of equation (I.) might have been expressed in a form similar to this, but upon examination it will be seen that it is reducible to the third given form (c_1).

It now remains only to find the solution of the Laplacian equation in the *third* or simple functional form, and this is given by

$$V = \left(\frac{d}{dz} - i\frac{d}{dx} - j\frac{d}{dy}\right)^{-1} \cdot \left(\frac{d}{dz} + i\frac{d}{dx} + j\frac{d}{dy}\right)^{-1} \cdot 0,$$

or is

$$V = \phi(x + iz,\ y + jz) + \psi(x - iz,\ y - jz) \ldots\ldots (c_2),$$

the coincidence of which with the first form is obvious. The necessity of *interpreting* all the results of this article, which are analytically complete, is sufficiently obvious, and the practical value of the results mainly depends on their susceptibility of such interpretation. The first form of solution is easily interpretable by the ordinary principles of triplets. As regards the two latter, the writer regrets his inability hitherto to satisfy the same demand, and would solicit the attention of those who may favour the present paper with a perusal, to this point.

3. Again, a general solution of the still higher equation of the same type

$$\frac{d^2W}{dx^2} + \frac{d^2W}{dy^2} + \frac{d^2W}{dz^2} + \frac{d^2W}{dw^2} = 0 \ldots\ldots (III.)$$

is given by $W = \Sigma A e^{m_1 x + m_2 y + m_3 z + m_4 w}$,

where m_1, m_2, m_3, m_4 are connected by the relation

$$m_1^2 + m_2^2 + m_3^2 + m_4^2 = 0.$$

Now, by the Calculus of *Quaternions*, this is equivalent to

$$(m_4 + im_1 + jm_2 + km_3)(m_4 - im_1 - jm_2 - km_3) = 0,$$

where i, j, and k are such operations that

$$\left.\begin{aligned} i^2 = j^2 = k^2 = -1 \\ ij = -ji,\quad jk = -kj,\quad ki = -ik \end{aligned}\right\},$$

and the solution becomes

$$W = \begin{cases} \Sigma A e^{m_1 x + m_2 y + m_3 z + (im_1 + jm_2 + km_3) w} \\ + \\ \Sigma B e^{m_1 x + m_2 y + m_3 z - (im_1 + jm_2 + km_3) w} \end{cases}.$$

To obtain the form of this corresponding to (a_1) and (a_2), we assume

$$m_1 = r\cos\alpha, \quad m_2 = r\cos\beta, \quad m_3 = r\cos\gamma,$$

and we have

$$W = \begin{cases} \Sigma A e^{m_1x + m_2y + m_3z + i_R . rw}, \\ + \\ \Sigma B e^{m_1x + m_2y + m_3z - i_R . rw}, \end{cases}$$

where

$$\left.\begin{aligned} i_R &= i\cos\alpha + j\cos\beta + k\cos\gamma \\ r^2 &= m_1^2 + m_2^2 + m_3^2 \end{aligned}\right\},$$

and a general solution of (III.) in the *first* form is

$$W = \left\{\begin{array}{l} \Sigma A e^{m_1x + m_2y + m_3z} \{\cos\sqrt{(m_1^2 + m_2^2 + m_3^2)}\, w \\ \qquad + i_R . \sin\sqrt{(m_1^2 + m_2^2 + m_3^2)}\, w\} \\ + \\ \Sigma B e^{m_1x + m_2y + m_3z} \{\cos\sqrt{(m_1^2 + m_2^2 + m_3^2)}\, w \\ \qquad - i_R . \sin\sqrt{(m_1^2 + m_2^2 + m_3^2)}\, w\} \end{array}\right\} \ldots (a_3),$$

and its duplicate

$$W = \left\{\begin{array}{l} \Sigma A e^{\sqrt{(m_1^2 + m_2^2 + m_3^2)}\, w} \{\cos(m_1x + m_2y + m_3z) \\ \qquad + i_R . \sin(m_1x + m_2y + m_3z)\} \\ + \\ \Sigma B e^{\sqrt{(m_1^2 + m_2^2 + m_3^2)}\, w} \{\cos(m_1x + m_2y + m_3z) \\ \qquad - i_R . \sin(m_1x + m_2y + m_3z)\} \end{array}\right\} \ldots (a_3').$$

Again, as before, regarding A and B as arbitrary functions of m_1, m_2, m_3, we can throw these solutions into the following shapes:

$$W = \left\{\begin{array}{l} \iiint \Phi(m_1, m_2, m_3)\, e^{m_1x + m_2y + m_3z} \\ \qquad \cos\sqrt{(m_1^2 + m_2^2 + m_3^2)}\, w \,.\, dm_1 dm_2 dm_3 \\ + \\ \iiint \Psi(m_1, m_2, m_3)\, e^{m_1x + m_2y + m_3z} \\ \qquad \sin\sqrt{(m_1^2 + m_2^2 + m_3^2)}\, w \,.\, dm_1 dm_2 dm_3 \end{array}\right\} \ldots (a_3)',$$

with its duplicate

$$W = \left\{\begin{array}{l} \iiint \Phi(m_1, m_2, m_3) \cos(m_1x + m_2y + m_3z) \\ \qquad e^{\sqrt{(m_1^2 + m_2^2 + m_3^2)}\, w} \,.\, dm_1 dm_2 dm_3 \\ + \\ \iiint \Psi(m_1, m_2, m_3) \sin(m_1x + m_2y + m_3z) \\ \qquad e^{\sqrt{(m_1^2 + m_2^2 + m_3^2)}\, w} \,.\, dm_1 dm_2 dm_3 \end{array}\right\} \ldots (a_3)'',$$

the limits of the integrals in both cases being supposed independent of the quantities x, y, z, and w. The same remark which has been made upon the similar solutions of (I.) and (II.) will apply here with equal force.

By a modification of Poisson's solution of the equation of oscillatory motion in an unlimited gas, we should obtain a solution of equation (III.) strictly in the second form, viz.:

$$4\pi W^* = \left\{ \begin{array}{l} \displaystyle\int_0^{\pi}\int_0^{2\pi} w\Phi(x + w\cos u\sqrt{-1},\quad y + w\sin u\cos v\sqrt{-1}, \\ \qquad\qquad z + w\sin u\sin v\sqrt{-1})\sin u\,du\,dv \\ \qquad + \\ \dfrac{d}{dw}\displaystyle\int_0^{\pi}\int_0^{2\pi} w\Psi(x + w\cos u\sqrt{-1},\quad y + w\sin u\cos v\sqrt{-1}, \\ \qquad\qquad z + w\sin u\sin v\sqrt{-1})\sin u\,du\,dv \end{array} \right\}$$

$\ldots\ldots\ldots (b_3).$

To complete the discussion of equation (III.), it now only remains to find its solution in the *third* or simple functional

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* By a regular deduction of the integral from the equation

$$W = \left(\frac{d}{dw} - i\frac{d}{dx} - j\frac{d}{dy} - k\frac{d}{dz}\right)^{-1}\cdot\left(\frac{d}{dw} + i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}\right)^{-1}\cdot 0,$$

or, putting

$$i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz} = D,$$

from the equation

$$2W = (e^{wD} - e^{-wD})\,\phi(xyz) + (e^{wD} + e^{-wD})\,\psi(xyz),$$

a form can be obtained, which seems to the writer to possess somewhat greater generality, viz. $4\pi W =$

$$\left\{ \begin{array}{l} \displaystyle\int_0^{\pi}\int_0^{2\pi} w\Phi(x + iw\cos u,\ y + jw\sin u\cos v,\ z + kw\sin u.\sin v)\sin u\,du\,dv \\ \qquad + \\ \dfrac{d}{dw}\displaystyle\int_0^{\pi}\int_0^{2\pi} w\Psi(x + iw\cos u,\ y + jw\sin u\cos v,\ z + kw\sin u.\sin v)\sin u\,du\,dv, \end{array} \right.$$

and by a similar process applied to the equation of the last article, its solution would be

$$4\pi V = \left\{ \begin{array}{l} \displaystyle\int_0^{\pi}\int_0^{2\pi} z\Phi(x + iz\cos u,\ y + jz\sin u\cos v)\sin u\,du\,dv \\ \qquad + \\ \dfrac{d}{dz}\displaystyle\int_0^{\pi}\int_0^{2\pi} z\Psi(x + iz\cos u,\ y + jz\sin u\cos v)\sin u\,du\,dv. \end{array} \right.$$

From the distinct geometrical characters which we are able to assign to the several symbols i, j, k, it would seem that their occurrence, so far from being matter of objection, will yet be found to possess some important bearing upon the problems, in whose solutions such symbols are exhibited.

form, which is

$$W = \phi(x + iw, y + jw, z + kw) + \psi(x - iw, y - jw, z - kw) \ldots (c_3),$$

and whose correspondence with the third forms of the solutions of equations (I.) and (II.), as well as its coincidence with its own first form, are both obvious.

4. It is evident that, by the adoption of an extended system of operations, regulated by laws similar to those already employed, the same methods of solution may be applied to the general equation containing n independent variables. However, when the number of operations indicated by the letters i, j, k, &c. exceeds *three*, we can no longer attach to them distinct geometrical conceptions. Thus the signification of i_8 is purely analytical, and on that account it seems unlikely that the examination of the general equation would lead to any practical result.

To adopt the language of Sir William Hamilton, the quantities i, i_7, i_8, i_8, &c. are, severally, *imaginary units;* that is, their moduli are positive unity and their squares negative unity. If for each of these quantities we had substituted the root of negative unity, the solutions in this form could have been obtained at once. Since however, as has been before remarked, it is probable that these several *units* bear some relation to the characters of the problems, whose laws are represented, respectively, by the equations (I.), (II.), (III.), it has been thought better to state the solutions in their present form. It will be noticed that the imaginary unit in the solution of (II.) has the same reference to an unit circle, as that in the solution of (III.) has to an unit sphere.

Again, it is to be observed that we are not at liberty to write the second form of the solution of (II.) in the shape

$$V = \begin{vmatrix} \Sigma A e^{m_1 x + m_2 y} (\cos m_1 z + i \,.\, \sin m_1 z)(\cos m_2 z + j \,.\, \sin m_2 z) \\ + \\ \Sigma B e^{m_1 x + m_2 y} (\cos m_1 z - i \,.\, \sin m_1 z)(\cos m_2 z - j \,.\, \sin m_2 z), \end{vmatrix}$$

as at first sight it might be supposed; and a similar remark will apply to the second forms of the general solution of (III.). In fact, neither in the Calculus of Triplets, nor in that of Quaternions, does the ordinary property of exponential functions, namely,

$$f(T) \,.\, f(T') = f(T + T'),$$
$$f(Q) \,.\, f(Q') = f(Q + Q'),$$

hold good, unless T and T', Q and Q', be respectively co-directional.* In general, it will be seen that the statement of the solution in the above form is equivalent to the breaking up of the *imaginary unit* into its constituents, and to the illegitimacy of this process the fact now mentioned reduces itself.

5. In conclusion, and as additional examples of the application of these imaginary symbols to integration, it is proposed to shew that by their use we can obtain, very simply, integrals of several well-known differential equations, and that they may be employed with advantage generally when the equations to be solved are symmetrical.

(I.) As a first example, let the solution of the familiar equation

$$dx^2 + dy^2 + dz^2 = ds^2,$$

which has recently occupied so much attention with French geometers, be sought.

It is evidently equivalent to

$$(dz + idx + jdy)(dz - idx - jdy) = ds^2,$$

and the solution is given by the system

$$\left.\begin{aligned} z + ix + jy &= \phi(s) + \gamma_1 \\ z - ix - jy &= \psi(s) + \gamma_2 \\ \phi'(s)\,\psi'(s) &= 1 \end{aligned}\right\},$$

where γ_1 and γ_2 are arbitrary vectors.

(II.) The general solution of the equation

$$\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 = \frac{1}{m^2}$$

is, by an exactly similar process, given by the system

$$\left.\begin{aligned} z + ix + jy &= \frac{1}{m}\Phi(s) + a_1 s + \beta_1 \\ z - ix - jy &= \frac{1}{m}\Psi(s) + a_2 s + \beta_2 \\ \Phi''(s)\,\Psi''(s) &= 1 \end{aligned}\right\},$$

where, as before, a_1, β_1, a_2, β_2 are arbitrary vectors.

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Thus, by the combination of the system of the previous example with that just stated, we obtain the general representation of the curve of double curvature, whose curvature is constant.

(III.) To find the integral of the equation of vibratory motion of thin plates, namely

$$\frac{d^2z}{dt^2} + b^2\left(\frac{d^4z}{dx^4} + 2\,\frac{d^4z}{dx^2.dy^2} + \frac{d^4z}{dy^4}\right) = 0.$$

Let $$i\,\frac{d}{dx} + j\,\frac{d}{dy} = D,$$

and the equation becomes

$$\frac{d^2z}{dt^2} + b^2D^4z = 0,$$

or $$z = e^{i_R bt D^2}.\,\phi(xy) + e^{-i_R bt D^2}.\,\psi(xy).$$

Now $$\int_{-\infty}^{\infty} dw\,.\,e^{-(w-x)^2} = \sqrt{\pi}\,;$$

and putting alternately

$$\alpha = \sqrt{(i_P\,.\,bt)}\left(i\,\frac{d}{dx} + j\,\frac{d}{dy}\right),$$

$$\alpha = \sqrt{(-i_P.bt)}\left(i\,\frac{d}{dx} + j\,\frac{d}{dy}\right),$$

we get for the required solution,

$$\sqrt{\pi}.z = \begin{cases} \displaystyle\int_{-\infty}^{\infty} dw.e^{-w^2}.\phi\{x + 2iw\sqrt{(i_P.\,bt)},\ y + 2jw\sqrt{(-i_P.\,bt)}\}, \\ \qquad + \\ \displaystyle\int_{-\infty}^{\infty} dw.e^{-w^2}.\psi\{x + 2iw\sqrt{(-i_P.bt},\ y + 2jw\sqrt{(-i_P.bt)}\}. \end{cases}$$

(IV.) By a similar process the integral of the equation

$$\frac{d^2v}{dt^2} + b^2\left(\frac{d^4v}{dx^4} + \frac{d^4v}{dy^4} + \frac{d^4v}{dz^4} + 2\,\frac{d^4v}{dy^2dz^2} + 2\,\frac{d^4v}{dz^2dx^2} + 2\,\frac{d^4v}{dx^2dy^2}\right) = 0$$

would be $\sqrt{\pi}v =$

$$\begin{cases} \displaystyle\int_{-\infty}^{\infty} dw.e^{-w^2}.\phi\{x + 2iw\sqrt{(i_R.bt},\ y + 2jw\sqrt{(i_R.bt)},\ z + 2kw\sqrt{(i_R.bt)}\}, \\ \qquad + \\ \displaystyle\int_{-\infty}^{\infty} dw.e^{-w^2}.\psi\{x + 2iw\sqrt{(-i_R.bt)},\ y + 2jw\sqrt{(-i_R.bt)},\ z + 2kw\sqrt{(-i_R.bt)}\}. \end{cases}$$

As regards the last two results, it may be again observed that they too demand interpretation, and that upon their susceptibility of such their practical value will depend.*

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ON SOME PROPERTIES OF A PARABOLA TOUCHING THE THREE SIDES OF A TRIANGLE.

By WILLIAM B. COLTMAN, Trinity College, Cambridge.

1. Let P be any point in the plane of a given triangle ABC. Let the sides of this triangle be produced indefinitely, and let α, β, γ be the lengths of the perpendiculars from P on BC, CA, AB. Then it is evident that if α, β, γ are known, the position of the point P will be determined. BC, CA, AB are called lines of reference, and α, β, γ the trilinear coordinates of the point P. The quantities α, β, γ are not independent of each other, for they are connected by an identical equation. Thus, let

$$BC = a, \quad CA = b, \quad AB = c, \quad \Delta = \text{area of } ABC,$$

then, if P lies within the triangle ABC, we have

$$a\alpha + b\beta + c\gamma = 2\Delta PBC + 2\Delta PCA + 2\Delta PAB$$
$$= 2\Delta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If P lies on the other side of one of the lines of reference, we must consider the corresponding perpendicular to be negative. Thus, if P lies on the other side of AB, we have

$$a\alpha + b\beta + c\gamma = 2\Delta PBC + 2\Delta PCA - 2\Delta PAB$$
$$= 2\Delta \text{ as before.}$$

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* After the present paper had been forwarded for publication, the writer became aware that Sir William Hamilton had called attention to the importance of the symbol

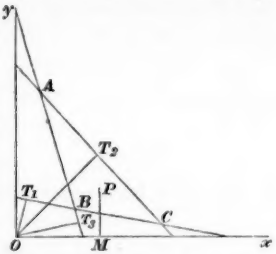
$$i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz},$$

in the *Proceedings of the Royal Irish Academy* for July, 1846, as also to the law of combination of this with the similar symbol

$$i\frac{d}{dx'} + j\frac{d}{dy'} + k\frac{d}{dz'},$$

in the *Philosophical Magazine* for October, 1847. The latter point seems, like the former, most valuable and likely to reward investigation.

2. We may also consider α, β, γ as functions of x and y. For let Ox, Oy be any lines at right angles to each other in the plane ABC. Let x, y be the coordinates of the point P referred to Ox, Oy as coordinate axes. Let OT_1, OT_2, OT_3 be perpendiculars on BC, CA, AB, respectively. Let



$$OT_1 = p_1,\ OT_2 = p_2\ OT_3 = p_3;$$
$$T_1Ox = \omega_1,\ T_2Ox = \omega_2,\ T_3Ox = \omega_3.$$

Then, taking P within the triangle ABC, in order that α, β, γ may all be positive, it is easy to see that we shall have

$$\alpha = x\cos\omega_1 + y\sin\omega_1 - p_1,$$
$$\beta = p_2 - x\cos\omega_2 - y\sin\omega_2,$$
$$\gamma = x\cos\omega_3 + y\sin\omega_3 - p_3,$$
$$\omega_1 - \omega_2 = C;\quad \omega_2 - \omega_3 = A;\quad \omega_1 - \omega_3 = \pi - B.$$

Also the equations to BC, CA, AB, will be respectively

$$\alpha = 0,\quad \beta = 0,\quad \gamma = 0.$$

3. The general equation to any straight line referred to trilinear coordinates may be put under the form

$$A_1\alpha + B_1\beta + C_1\gamma = 0.$$

For it is a linear function of x and y, and contains two arbitrary constants. It is therefore the most general form of the equation to the straight line. If a curve of the second order be inscribed in the triangle ABC its equation will be

$$u = l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta = 0 \ldots (2).$$

For the points at which AB meets (2) are got by combining (2) with the equation $\gamma = 0$, whence we get

$$0 = (l\alpha - m\beta)^2,$$

which shews that the two points of intersection coincide; that is, that the curve touches AB. Similarly we may shew that it touches AC and BC. Also it is of the second degree. For it is of the second degree in α, β, γ, which are linear functions of x and y. Mr. Salmon has discussed the properties of this equation when the curve which it represents is any conic section. The object of this paper is to establish certain equations which obtain when the curve is a parabola. It is however necessary first to prove some preliminary propositions.

PROP. 1. To find the length of the perpendicular from any point $(\alpha'\beta'\gamma')$ on the line $A_1\alpha + B_1\beta + C_1\gamma = 0$. The length of a perpendicular from a point $x'y'$ on a line $(a_1x + b_1y = c_1)$ is $(a_1x' + b_1y' - c_1)(a_1^2 + b_1^2)^{-\frac{1}{2}}$. Substituting for α, β, γ, their values in terms of x and y, we find that in the present case

$$\begin{aligned} a_1^2 + b_1^2 &= (A_1\cos\omega_1 - B_1\cos\omega_2 + C_1\cos\omega_3)^2 + (A_1\sin\omega_1 - B_1\sin\omega_2 + C_1\sin\omega_3)^2 \\ &= A_1^2 + B_1^2 + C_1^2 - 2B_1C_1\cos(\omega_2 - \omega_3) + 2C_1A_1\cos(\omega_3 - \omega_1) - 2A_1B_1\cos(\omega_1 - \omega_2) \\ &= A_1^2 + B_1^2 + C_1^2 - 2B_1C_1\cos A - 2C_1A_1\cos B - 2A_1B_1\cos C. \end{aligned}$$

Therefore the length of the perpendicular

$$= \frac{A_1\alpha' + B_1\beta' + C_1\gamma'}{\sqrt{(A_1^2 + B_1^2 + C_1^2 - 2B_1C_1\cos A - 2C_1A_1\cos B - 2A_1B_1\cos C)}} \quad \ldots\ldots\ldots\ldots (3).$$

COR. The perpendiculars on two lines from any point of a line bisecting the angle between them are equal. Hence, if the equations to the lines are

$$A_1\alpha + B_1\beta + C_1\gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$A_2\alpha + B_2\beta + C_2\gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

the equations to their internal and external bisectors will be

$$\frac{A_1\alpha + B_1\beta + C_1\gamma}{\sqrt{(A_1^2+B_1^2+C_1^2-2B_1C_1\cos A-2C_1A_1\cos B-2A_1B_1\cos C)}} = \pm \frac{A_2\alpha + B_2\beta + C_2\gamma}{\sqrt{(A_2^2+B_2^2+C_2^2-2B_2C_2\cos A-2C_2A_2\cos B-2A_2B_2\cos C)}}.$$

PROP. 2. To find the condition that the two lines (4) and (5) may include an angle ϕ.

Proceeding as in the former proposition, we find that we must have

$$\tan\phi = \pm \frac{A_1(B_2c - C_2b) + B_1(C_2b - A_2c) + C_1(A_2b - B_2a)}{A_1(A_2 - B_2\cos C - C_2\cos B) + B_1(B_2 - C_2\cos A - A_2\cos C) + C_1(C_2 - A_2\cos B - B_2\cos A)}.$$

If (4) and (5) are parallel $\tan\phi = 0$, if they are perpendicular $\tan\phi = \infty$, in which cases we have respectively

$$0 = A_1(B_2c - C_2b) + B_1(C_2b - A_2c) + C_1(A_2b - Ba) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and $0 = A_1(A_2 - B_2\cos C - C_2\cos B) + B_1(B_2 - C_2\cos A - A_2\cos C) + C_1(C_2 - A_2\cos B - B_2\cos A) \ldots (7).$

PROP. 3. To find the equation to a tangent to any curve referred to trilinear coordinates.

Let the line (4) cut the curve $v = 0$ in the points $(\alpha'\beta'\gamma')(\alpha''\beta''\gamma'')$, then we have

$$A_1\alpha' + B_1\beta' + C_1\gamma' = 0, \qquad A_1\alpha'' + B_1\beta'' + C_1\gamma'' = 0.$$

Therefore (eliminating A_1, B_1, and C_1, by cross-multiplication, from these equations and from (4)) we get

$$\alpha(\beta'\gamma'' - \gamma'\beta'') + \beta(\gamma'\alpha'' - \alpha'\gamma'') + \gamma(\alpha'\beta'' - \beta'\alpha'') = 0,$$

or $$\alpha\{\beta'(\gamma'' - \gamma') - \gamma'(\beta'' - \beta')\} + \beta\{\gamma'(\alpha'' - \alpha') - \alpha'(\gamma'' - \gamma')\} + \gamma\{\alpha'(\beta'' - \beta') - \beta'(\alpha'' - \alpha')\} = 0.$$

When the secant becomes a tangent, this equation becomes

$$\alpha(\beta'd\gamma' - \gamma'd\beta') + \beta(\gamma'd\alpha' - \alpha'd\gamma') + \gamma(\alpha'd\beta' - \beta'd\alpha') = 0, \quad \text{or } (\beta\gamma' - \gamma\beta')\,d\alpha' + (\gamma\alpha' - \alpha\gamma')\,d\beta + (\alpha\beta' - \beta\alpha')\,d\gamma = 0.$$

Also differentiating (1) and the equation to the curve, we get

$$ad\alpha' + bd\beta' + cd\gamma' = 0 \text{ and } \frac{dv}{d\alpha'}d\alpha' + \frac{dv}{d\beta'}d\beta' + \frac{dv}{d\gamma'}d\gamma' = 0.$$

Eliminating $d\alpha'$, $d\beta'$, $d\gamma'$, by cross-multiplication, we have

$$0 = \frac{dv}{d\alpha'}\{c(\gamma\alpha' - \alpha\gamma') + b(\beta\alpha' - \alpha\beta')\} + \frac{dv}{d\beta'}\{a(\alpha\beta' - \beta\alpha') + c(\gamma\beta' - \beta\gamma')\} + \frac{dv}{d\gamma'}\{b(\beta\gamma' - \gamma\beta') + a(\alpha\gamma' - \gamma\alpha')\},$$

$$= \frac{dv}{d\alpha'}\{\alpha'(b\beta + c\gamma) - \alpha(b\beta' + c\gamma')\} + \frac{dv}{d\beta'}\{\beta'(c\gamma + a\alpha) - \beta(c\gamma' + a\alpha')\} + \frac{dv}{d\gamma'}\{\gamma'(a\alpha + b\beta) - \gamma(a\alpha' + b\beta')\},$$

$$= \frac{dv}{d\alpha'}\{\alpha' - \alpha\}\,2\Delta + \frac{dv}{d\beta'}2\Delta(\beta' - \beta) + \frac{dv}{d\gamma'}2\Delta(\gamma' - \gamma) \quad \text{by (1).}$$

But since v is a homogeneous function of α, β, γ, $\alpha'\frac{dv}{d\alpha'} + \beta'\frac{dv}{d\beta'} + \gamma'\frac{dv}{d\gamma'} = 0$, and the equation to the tan-

Note.—If there are n lines of reference $a_1a_2a_3 \ldots a_n$ the equation to the tangent will be

$$a_1\frac{dv}{da_1'} + a_2\frac{dv}{da_2'} + \ldots + a_n\frac{dv}{da_n'} = 0.$$

gent becomes

$$\alpha \frac{dv}{d\alpha'} + \beta \frac{dv}{d\beta'} + \gamma \frac{dv}{d\gamma'} = 0 \ldots\ldots\ldots (8).$$

Cor. 1. The equation to the normal may be found at once from the general equation to the straight line by introducing the conditions, that it passes through the point $\alpha'\beta'\gamma'$, and that it is perpendicular to the tangent.

Cor. 2. Let α, β, γ be a fixed point. Then it is easy to see from (8) that the points of contact of all the tangents which can be drawn through the point $\alpha\beta\gamma$ to touch the curve $v = 0$ lie on the curve

$$\alpha \frac{dv}{d\alpha'} + \beta \frac{dv}{d\beta'} + \gamma \frac{dv}{d\gamma'} = 0 \ldots\ldots\ldots (9),$$

where $\alpha'\beta'\gamma'$ are supposed to be variable. If $v = 0$ is of the second degree this equation becomes linear in $\alpha'\beta'\gamma'$, and is evidently the equation to the chord joining the points of contact of the two tangents through the point $\alpha\beta\gamma$. This point is called the pole of the chord of contact, and the chord of contact is called the polar of this point.

Cor. 3. The equation (2) may be put under the form $(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} = 0$. Hence the equation to a tangent to it is

$$\left(\frac{l}{\alpha'}\right)^{\frac{1}{2}}\alpha + \left(\frac{m}{\beta'}\right)^{\frac{1}{2}}\beta + \left(\frac{n}{\gamma'}\right)^{\frac{1}{2}} = 0.$$

Therefore if the line (4) touches (2), we must have

$$kl^{\frac{1}{2}} = A_1\alpha'^{\frac{1}{2}}, \quad km^{\frac{1}{2}} = B_1\beta'^{\frac{1}{2}}, \quad kn^{\frac{1}{2}} = C_1\gamma'^{\frac{1}{2}},$$

where k is indeterminate. Multiplying these equations by $A_1^{-1}l^{\frac{1}{2}}$, $B_1^{-1}m^{\frac{1}{2}}$, $C_1^{-1}n^{\frac{1}{2}}$, respectively, and adding, we have

$$\frac{l}{A} + \frac{m}{B} + \frac{n}{C} = 0 \ldots\ldots\ldots\ldots (10),$$

as the condition that (4) may touch (2).

4. We may now proceed to find the condition that (2) may be a parabola. From (1) and (2) we have, eliminating α,

$$0 = l^2(2\Delta - b\beta - c\gamma)^2 + a^2(m^2\beta^2 + n^2\gamma^2) - 2mna^2\beta\gamma - 2la(m\beta + n\gamma)(2\Delta - b\beta - c\gamma).$$

Now if AB, AC be taken as coordinate axes, β and γ will be proportional to the coordinates of a point in the curve. Hence, that the curve may be a parabola, we must have

(coefficient of $\beta\gamma)^2$ = 4 (coefficient of β^2) (coefficient of γ^2),

or $4(l^2bc + lmac + lnba - mna^2)^2 = 4(lb + ma)^2 - (lc + na)^2$
$= 4(l^2bc + lmac + lnab + mna^2)^2,$

or $0 = lmn(lbc + mac + nba).$

Now if either l, m, or n become zero, (2) becomes the equation to two straight lines. Hence the condition that (2) may be a parabola is

$$\frac{l}{a} + \frac{m}{b} + \frac{n}{c} = 0 \ldots\ldots\ldots\ldots\ldots (11).$$

Cor. 1. Comparing (10) and (11) we see that the geometrical property expressed by (11) is that the parabola (2) will touch the line $a\alpha + b\beta + c\gamma = 0$, which is a line at infinity.

Cor. 2. By similar reasoning we find, that if the general trilinear equation of the second order

$$A_1\alpha^2 + B_1\beta^2 + C_1\gamma^2 + A'\beta\gamma + B'\gamma\alpha + C'\alpha\beta = 0$$

represents a circle, we must have

$$B_1c^2 + C_1b^2 - A'bc = C_1a^2 + A_1c^2 - B'ac = A_1b^2 + B_1a^2 - C'ba;$$

and that the conditions that the curves

$$l\beta\gamma + m\alpha\gamma + n\alpha\beta = 0, \quad l^2\alpha^2 + m^2\beta^2 = n^2\gamma^2, \quad \alpha\gamma = k\beta^2$$

may be parabolæ, are respectively

$$(al)^{\frac{1}{2}} + (bm)^{\frac{1}{2}} + (cn)^{\frac{1}{2}} = 0, \quad \frac{a^2}{l} + \frac{b^2}{m} = \frac{c^2}{n}, \quad b^2 = 4kac.$$

The geometrical interpretation of the last three equations is similar to that of the equation (11).

5. To find the equations to the focus, director, and axis of the parabola (2).

Let the line (4) touch the curve (2) and be at right angles to AB. Then, from (7), we have

$$C_1 = A_1 \cos B + B_1 \cos A \ldots\ldots\ldots (12),$$

and, from (10) and (11), we have, by eliminating n,

$$A_1\frac{a}{l}(cB_1 - C_1b) = B_1\frac{b}{m}(aC_1 - cA_1). \ldots\ldots (13).$$

Now from (12)

$$aC_1 - cA_1 = A_1a\cos B + aB_1\cos A - cA_1$$
$$= A_1(a\cos B - c) + aB_1\cos A$$
$$= (aB_1 - bA_1)\cos A.$$

Similarly $\quad cB_1 - bC_1 = (aB_1 - bA_1)\cos B.$

Therefore, from (13),

$$\frac{l}{a}\frac{\cos A}{A_1} = \frac{m}{b}\frac{\cos B}{B_1} \ldots\ldots\ldots (14).$$

Let $\alpha_1, \beta_1, \gamma_1$, be the coordinates of the focus. By a property of the parabola we know that the polar of any point in which two tangents at right angles to each other intersect, passes through the focus. Hence the polar of the point $\{\gamma = 0,\ A_1\alpha + B_1\beta = 0\}$ passes through the focus, therefore $\alpha_1, \beta_1, \gamma_1$, satisfy its equation, which is by (9)

$$\frac{1}{A_1}\frac{du}{d\alpha} = \frac{1}{B_1}\frac{du}{d\beta}.$$

Therefore, from (14),

$$\frac{a}{l\cos A}\frac{du}{d\alpha_1} = \frac{b}{m\cos B}\frac{du}{d\beta_1},$$

or $\dfrac{l\alpha_1 - m\beta_1 - n\gamma_1}{a^2 - b^2 - c^2} = \dfrac{m\beta_1 - n\gamma_1 - l\alpha_1}{b^2 - c^2 - a^2} = \dfrac{n\gamma_1 - l\alpha_1 - m\beta_1}{c^2 - a^2 - b^2}$ by symmetry,

or $$\frac{l\alpha_1}{a^2} = \frac{m\beta_1}{b^2} = \frac{n\gamma_1}{c^2} = \lambda \ldots\ldots\ldots\ldots (15),$$

which are the equations to the focus. Also the director is the polar of the focus. Hence, by (9), its equation is

$$0 = \alpha_1\frac{du}{d\alpha} + \beta_1\frac{du}{d\beta} + \gamma_1\frac{du}{d\gamma}$$

and, by (15),

$$= a^2(l\alpha - m\beta - n\gamma) + b^2(m\beta - n\gamma - l\alpha) + c^2(n\gamma - l\alpha - m\beta)$$
$$= (a^2 - b^2 - c^2)\, l\alpha + m\beta\,(b^2 - c^2 - a^2) + n\gamma\,(c^2 - a^2 - b^2),$$

or $$0 = \frac{l\alpha}{a}\cos A + \frac{m\beta}{b}\cos B + \frac{n\gamma}{c}\cos C \ldots\ldots\ldots\ldots (16).$$

Let $\alpha_2\beta_2\gamma_2$ be the focus at infinity. Then, by a general property of conic sections,

$$\alpha_1\alpha_2 = \beta_1\beta_2 = \gamma_1\gamma_2.$$

Therefore, by (15),

$$\frac{a^2\alpha_2}{l} = \frac{b^2\beta_2}{m} = \frac{c^2\gamma_2}{n} \ldots\ldots\ldots\ldots (17).$$

Let the equation to the axis be

$$A_1\alpha + B_1\beta + C_1\gamma = 0.$$

Since $\alpha_1\beta_1\gamma_1$ is a point in the axis,

$$A_1\frac{a^2}{l} + B_1\frac{b^2}{m} + C_1\frac{c^2}{n} = 0,$$

and since $\alpha_2\beta_2\gamma_2$ is also a point in the axis,

$$A_1\frac{l}{a^2} + B_1\frac{m}{b^2} + C_1\frac{n}{c^2} = 0.$$

(This last condition may also be found from the condition that the axis is at right angles to the director.) Eliminating A_1, B_1, C_1, we have as the equation to the axis

$$\frac{\alpha a^2}{l}\left(\frac{b^4}{m^2}-\frac{c^4}{n^2}\right)+\frac{\beta b^2}{m}\left(\frac{c^4}{n^2}-\frac{a^4}{l^2}\right)+\frac{\gamma c^2}{n}\left(\frac{a^4}{l^2}-\frac{b^4}{m^2}\right)=0.$$

6. To find the magnitude of the latus-rectum L.

$\frac{1}{2}L$ = the length of the perpendicular from the focus on the director. Therefore, from (3) and (16),

$$\frac{1}{2}L=\frac{\alpha_1\frac{l}{a}\cos A+\beta_1\frac{m}{b}\cos B+\gamma_1\frac{n}{c}\cos C}{\sqrt{\left\{\frac{l^2}{a^2}\cos^2 A+\frac{m^2}{b^2}\cos^2 B+\frac{n^2}{c^2}\cos^2 C-2\cos A\cos B\cos C\left(\frac{lm}{ab}+\frac{mn}{bc}+\frac{nl}{ac}\right)\right\}}}$$

or, from (11) and (15),

$$=\frac{(a\cos A+b\cos B+c\cos C)\,\lambda}{\sqrt{\left\{\frac{l^2}{a^2}\cos A(\cos A+\cos B\cos C)+\frac{m^2}{b^2}\cos B(\cos B+\cos C\cos A)+\frac{n^2}{c^2}\cos C(\cos C+\cos A\cos B)\right\}}}.$$

But $a\cos A+b\cos B+c\cos C=2a(\cos A+\cos B\cos C)$; therefore

$$\frac{L}{2}=\frac{\lambda\sqrt{\{2(a\cos A+b\cos B+c\cos C)(abc)^3\}}}{\sqrt{(l^2b^3c^3\cos A+m^2c^3a^3\cos B+n^2a^3b^3\cos C)}};$$

and, from (1) and (15), $\lambda\left(\frac{a^3}{l}+\frac{b^3}{m}+\frac{c^3}{n}\right)=2\Delta$; therefore, if $\mu a=\sin A$,

$$L=(4\Delta)^{\frac{3}{2}}\{\mu(l^2b^3c^3\cos A+m^3c^3a^3\cos B+n^2a^3b^3\cos C)\}^{-\frac{1}{2}}\left(\frac{a^3}{l}+\frac{b^3}{m}+\frac{c^3}{n}\right)^{-1}.$$

7. By reasoning similar to the above, we may shew that the equations to the focus, director, and axis of the parabola whose equation is $4ac\alpha\gamma = b^2\beta^2$, are respectively

$$\frac{\alpha_1}{a} = \frac{b\beta_1}{2ac}\sec B = \frac{\gamma_1}{c}; \quad c\alpha - b\cos B\beta + a\gamma = 0,$$

$$2ac\{\alpha(a\cos B + c) - (a + c\cos B)\gamma\} = b\beta(a^2 - c^2),$$

and that its latus-rectum $= (4\Delta)^2 (b^2 + 2ac\cos B)^{-\frac{3}{2}}$.

8. The following problems serve to illustrate the use of the equations given above.

Prob. 1. The foci of all the parabolæ which can be escribed* to a triangle lie on the circle circumscribed about the triangle, and their directors will all pass through the point of intersection of perpendiculars from the angles on the opposite sides.

By (11) and (15) we have as the equation to the locus of the focus $\frac{a}{\alpha_1} + \frac{b}{\beta_1} + \frac{c}{\gamma_1} = 0$, the equation to the circle circumscribing ABC.

Also from (11) and (16) we see that the equation to the director may be put under the form

$$\frac{l}{a}(\rho - \alpha\cos A) + \frac{m}{b}(\rho - \beta\cos B) + \frac{n}{c}(\rho - \gamma\cos C) = 0,$$

and therefore the director passes through the point

$$\rho = \alpha\cos A = \beta\cos B = \gamma\cos C,$$

which is the point in which the perpendiculars from A, B, and C on BC, CA, AB intersect.

Prob. 2. Let S be the focus of the parabola (2). Join SA, SB, SC, and draw AA', AB', AC' at right angles to them respectively; AA', BB', CC' meet in a point.

The equation to SA is

$$\frac{m}{b^2}\beta = \frac{n}{c^2}\gamma.$$

Therefore, by (7), the equation to AA' is

$$\left(\frac{m}{b^2}\cos A + \frac{n}{c^2}\right)\beta + \left(\frac{n}{c^2}\cos A + \frac{m}{b^2}\right)\gamma = 0;$$

* [That is, touching one side of a triangle and the two others produced.]

or, by (11),

$$\left(\frac{n}{c^2}\cos C - \frac{l}{a^2}\cos A\right)\frac{\beta}{b} = \left(\frac{l\cos A}{a^2} - \frac{m\cos B}{b^2}\right)\frac{\gamma}{c}.$$

By symmetry the equations to BB', CC' are

$$\left(\frac{l}{a^2}\cos A - \frac{m}{b^2}\cos B\right)\frac{\gamma}{c} = \left(\frac{m\cos B}{b^2} - \frac{n\cos C}{c^2}\right)\frac{\alpha}{a};$$

$$\left(\frac{m}{b^2}\cos B - \frac{n}{c^2}\cos C\right)\frac{\alpha}{a} = \left(\frac{n\cos C}{c^2} - \frac{l\cos A}{a^2}\right)\frac{\beta}{b}.$$

From the forms of their equations AA', BB', CC' evidently meet in a point.

Prob. 3. Let a parabola touch BC, CA, AB in D, E, F, respectively. Join AD, BE, CF, and let them cut the curve in L, M, N. Let LP, MQ, NR be tangents to the curve, and let them meet BC, CA, AB in P, Q, R. Then $A'D$, $B'C$, $C'F$ meet in a point, the locus of which, for different parabolæ, is an ellipse circumscribing ABC. Also LP, MQ, NR, and the polars of A, B, C, all pass through fixed points, and the points P, Q, R lie on a straight line passing through the centre of gravity of ABC.

The equations to the point D are $(\alpha = 0, n\gamma = m\beta)$, therefore the equation to AD is $n\gamma = m\beta$. Similarly, the equations to BE and CF are $l\alpha = n\gamma$ and $m\beta = l\alpha$; therefore AD, BE, CF meet in a point $\rho = l\alpha = m\beta = n\gamma$. Therefore, by (11), the equation to its locus is $\frac{1}{a\alpha} + \frac{1}{b\beta} + \frac{1}{c\gamma} = 0$, the equation to an ellipse circumscribing ABC. The equation to the polar of A is, by (9),

$$l\alpha - m\beta - n\gamma = 0,$$

which, by (10), may be put under the form

$$l\left(\alpha - \frac{\rho}{a}\right) - m\left(\beta + \frac{\rho}{b}\right) - n\left(\gamma + \frac{\rho}{c}\right) = 0.$$

Therefore the polar of A passes through the fixed point $\rho = a\alpha = -b\beta = -c\gamma$.

Similarly, the polars of B and C pass through fixed points. Again, the equations to the point L are

$$\left\{(n\gamma)^{\frac{1}{2}} = (m\beta)^{\frac{1}{2}} = -\frac{\sqrt{l\alpha}}{2}\right\};$$

therefore, by (8), the equation to LP is $\frac{1}{2}l\alpha = m\beta + n\gamma$, which

by (11) may be put in the form

$$\tfrac{1}{2}l\left(\alpha - \frac{2\rho}{a}\right) = m\left(\beta + \frac{\rho}{b}\right) + n\left(\gamma + \frac{\rho}{c}\right).$$

Therefore LP passes through the fixed point

$$\rho = \tfrac{1}{2}a\alpha = -b\beta = -c\gamma.$$

Similarly, MQ and NR pass through fixed points.

Also the equations to P, Q, R are

$$\{\alpha = 0,\ m\beta + n\gamma = 0\},\ \{\beta = 0,\ n\gamma + l\alpha = 0\},\ \{\gamma = 0,\ l\alpha + m\beta = 0\};$$

therefore P, Q, R lie on a line whose equation is

$$l\alpha + m\beta + n\gamma = 0.$$

This equation, by (11), may be put under the form

$$\frac{l}{a}(\rho - a\alpha) + \frac{m}{b}(\rho - b\beta) + \frac{n}{c}(\rho - c\gamma) = 0;$$

therefore PQR passes through the point $\rho = a\alpha = b\beta = c\gamma$, which is the centre of gravity of the triangle ABC.

December 23, 1851.

Postscript.—Since this paper was sent to the Editor, Mr. Salmon's work *On the Higher Plane Curves* has been published. In it Mr. Salmon has established the equation (8) by a different process.

ON THE KNIGHT'S MOVE AT CHESS.

By Ferdinand Minding (of Dorpat).

[From the *Bulletin de l'Académie des Sciences de St. Petersburg*, tom. vi. No. 14. (lu le 22 Janvier 1847.)]

This well-known problem was first treated scientifically by Euler in the *Memoirs of Berlin* of the year 1759, where he shews a simple method of finding a great number of solutions, particularly of recurring circuits, of the Knight's move ("von wiederkehrenden Umläufen des Springers"). This method consists essentially in first occupying a series of squares till the Knight can move no farther without recurring to a square it had already occupied; the squares thus preoccupied he then endeavours to arrange in groups, which in an altered order may join each other as well as the squares not yet touched. It is, however, not my object

to explain this method here, which is the less required, as it is fully done in Legendre's *Théorie des Nombres*, vol. II. p. 151, of the 2nd edition; (also in Klügel and Mollweide's *Mathematical Dictionary*, in the article, "Springer auf dem Schachbrette.")

In the *Memoirs of the Paris Academy* of 1771, Vandermonde (Remarques sur les problèmes de situation) justly draws attention to the importance of researches in geometry of position. He compares the course of the Knight with that of a thread, which, wrapped round a pin placed in the centre of each square, connects all squares according to the law assigned to the Knight's move; but he confines himself to the investigation of particular symmetrical arrangements. In the above-mentioned Dictionary there is also quoted a memoir by Colini (Solution du problème du cavalier au jeu des échecs, Mannheim 1773), which contains several solutions; but I have not seen the book.

Legendre (p. 165 of the above work) touches upon the question relating to the number of all possible solutions, and this indeed is the principal question which it is the province of analysis to answer. But as this distinguished analyst designates the problem as a difficult one, and has recourse to a preliminary solution, which clearly he himself did not consider satisfactory, the necessity arose of inquiring more accurately into the nature of this difficulty, and I beg permission to lay before the reader the results of my researches. I have, indeed, not arrived at an actual numerical result, or rather at a knowledge of the means, by which such a result, if required, could in all cases be obtained; but what follows will shew that the only obstacle in the way is the almost unlimited extent of a calculation that leads to very many and very large numbers. It would be a valuable acquisition to overcome this impediment by appropriate approximations, as has been done successfully in other, though, it would seem, less complicated cases.

I denote the squares of the chess-board by numbers which may be regarded as the coordinates of their centres, and I make a corner square the origin. For this there is $x = 0$, $y = 0$; for the one diagonally opposite, $x = 7$, $y = 7$. The square for which $x = a$, $y = b$ is denoted by (a, b): of course the order of these letters has to be attended to. The Knight may pass in one move from (a, b) to all those squares that are denoted by $(a \pm 2, b \pm 1)$ and $(a \pm 1, b \pm 2)$, provided their coordinates do not exceed the limits 0 and 7. I further affix to each square an index, *i.e.* one of the numbers from 1 to

64, in a perfectly arbitrary order, but each index corresponds to but one square.

Let α be the index of any square; β, γ, δ, &c. the indices of all those, which the Knight may reach in one move, beginning from α; I shall call these the *neighbouring fields* (which, consequently, do not mean the adjacent ones). If now the Knight is on a particular square A, the question arises, in how many different ways he may reach the square α by n moves, allowing him to touch the same square repeatedly. Let $w_n^{(\alpha)}$ denote this number. If we separate the squares into even and odd (black and white) ones, α must be of the same kind with A, or of the opposite kind, according as the number of moves n is even or odd. Whenever this condition is not satisfied, $w_n^{(\alpha)} = 0$.

Since the Knight can reach α only from β, γ, δ, &c., we have immediately

$$w_n^{(\alpha)} = w^{(\beta)}{}_{n-1} + w^{(\gamma)}{}_{n-1} + w^{(\delta)}{}_{n-1} + \text{&c.} \ldots\ldots\ldots \text{(A)},$$

which reduces the n^{th} move to the $(n-1)^{\text{th}}$. Each square furnishes us a similar equation by giving to α successively the values 1, 2...64. Now we know the values of $w_n^{(\alpha)}$ for the first move, or for $n = 1$; this number being = 1 for all the neighbouring squares of A, and = 0 for all others. Hence we can calculate $w_n^{(\alpha)}$ by means of the 64 equations for every square and every number of moves. Thus the problem is solved.

The same problem may be solved under the additional condition, that any particular square shall not be touched. If β is such an excluded square, then $w_n^{(\beta)} = 0$ for all values of n, and at the same time in the system (A) that equation disappears which expresses the transition from the square A to β; hence we have in (A) one equation and one unknown quantity less. In general, as many equations and as many unknown quantities will disappear from the system of equations (A) as the number of excluded squares amounts to; and the number of equations (A) and of the unknown quantities they contain is in all cases equal to the number of admissible (open) fields, which we shall call i. Let indices 1, 2...i be assigned to them; α being one of them assigned to the square (a, b), let us form the product $w_n^{(\alpha)} x^a y^b$ for it, and similarly for all other open squares, and let U_n denote the sum of all these products, so that

$$U_n = \Sigma w_n^{(\alpha)} x^a y^b \text{ from } \alpha = 1 \text{ to } \alpha = i$$

comprises all cases that are possible in n moves. To pass from this to the $(n+1)^{\text{th}}$ move, U_n has to be multiplied with

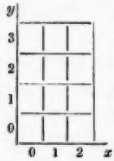
a factor U of a form related to the Knight's move, namely with

$$U = \left(x + \frac{1}{x}\right)\left(y^2 + \frac{1}{y^2}\right) + \left(x^2 + \frac{1}{x^2}\right)\left(y + \frac{1}{y}\right),$$

omitting in the product all terms in which negative powers of x or y and positive powers exceeding the 7th occur; also those that refer to excluded squares. These omissions I denote by inclosing $U_n U$ in square brackets, and call $[U_n U]$ an abridged *product*. Thus we get

$$U_{n+1} = \Sigma w^{\alpha}_{n+1} x^a y^b = [U_n U],$$

an equation which is nothing but a convenient form for embracing the equations (A). An example may serve to indicate some means of shortening the calculation.



In how many different ways is it possible, on a board containing three times four squares, to pass from a corner (0, 0) to the diagonally opposite one (2, 3) in 11 moves, if it is not allowed to touch again the starting-point (0, 0), nor to reach (2, 3) before the last (11th) move?

In this case (0, 0) is excluded altogether and (2, 3) from the first 10 moves. The abridged products are

$$U_1 = [U] = x^2y + xy^2$$

$$U_2 = [U_1 U] = x^2 + y^2 + xy^3$$

$$U_3 = [U_2 U] = x + 2y + 2x^2y + xy^2$$

$$U_4 = [U_3 U] = 3x^2 + 3y^2 + 3x^2y^2 + 4xy^3$$

$$U_5 = [U_4 U] = 6x + 10y + 7x^2y + 3xy^2 + 3y^3$$

$$U_6 = [U_5 U] = 13x^2 + 3xy + 13y^2 + 19x^2y^2 + 17xy^3.$$

Instead of continuing in this manner, we may start from (2, 3) and develop a series of five moves, that is to meet the former. Denoting in this case the abridged products by V_1, V_2, &c., we get

$$V_1 = [x^2y^3 U] = xy + y^2.$$

On account of the symmetry, it is unnecessary to calculate V; for V_n is found from U_n by changing x^a into x^{2-a} and y^b into y^{3-b}; hence we have at once

$$V_5 = 3x^2 + 3xy + 7y^2 + 10x^2y^2 + 6xy^3.$$

Hence since, starting from (0, 0), we can reach (2, 0) in 13 different ways by means of 6 moves, and by 5 moves in

3 different ways pass from (2, 3) to (2, 0) or from (2, 0) to (2, 3), there are $13 \times 3 = 39$ ways of passing in 11 moves from (0, 0) to (2, 3) under the restrictions of the problem, and occupying the square (2, 0) at the sixth move. If we apply this reasoning to all squares that may be reached in the same sixth move or all that occur in U_6, we get the total number C of ways in which it is possible to reach (2, 3) in the eleventh move, starting from (0, 0) and not touching this square again: we get

$$C = 13.3 + 3.3 + 13.7 + 19.10 + 17.6 = 431.$$

The closer investigation of the numbers $w_n^{(\alpha)}$ leads to certain recurring series, of which they are the terms.

We have, according to (A),

$$\begin{aligned} w^{(\alpha)}_{n+1} &= w_n^{(\beta)} + w_n^{(\gamma)} + w_n^{(\delta)} + \&c. \\ w_n^{(\alpha)} &= w^{(\beta)}_{n-1} + w^{(\gamma)}_{n-1} + w^{(\delta)}_{n-1} + \&c. \\ &\vdots \\ w^{(\alpha)}_{n+1-i} &= w^{(\beta)}_{n-i} + w^{(\gamma)}_{n-i} + w^{(\delta)}_{n-i} + \&c. \end{aligned}$$

Multiplying these by $1, \varepsilon_1, \varepsilon_2 \ldots \varepsilon_i$ in order, adding and putting

$$w_n^{(\alpha)} + \varepsilon_1 w^{(\alpha)}_{n-1} + \varepsilon_2 w^{(\alpha)}_{n-2} + \ldots + \varepsilon_i w^{(\alpha)}_{n-1} = W_n^{(\alpha)},$$

we get $$W^{(\alpha)}_{n+1} = W_n^{(\beta)} + W_n^{(\gamma)} + W_n^{(\delta)} + \&c. \ldots . (B),$$

an equation which represents i equations, arising from it by putting α successively $= 1, 2 \ldots i$, and then on the right-hand side in each case the indices of the neighbouring squares corresponding to each value of α. If now we determine the i coefficients $1, \varepsilon_1 \ldots \varepsilon_i$ by as many equations,

$$W^{(\alpha)}_{i+1} = 0 \quad \text{for } \alpha = 1, 2 \ldots i,$$

we get, by (B),

$$W^{(\alpha)}_{i+2} = 0, \text{ and generally } W_n^{(\alpha)} = 0,$$

for every $n > i$ and every α from 1 to i.

Thus the values of $w_n^{(\alpha)}$ for the several values of n form a recurring series, the scale of which is the same for all values of α. Hence they may also be considered as the coefficients of the development of i rational abgebraic fractions, all of which have the same denominator.

If t denotes an indeterminate quantity, this denominator is

$$\psi t = t^i + \varepsilon_1 t^{i-1} + \varepsilon_2 t^{i-2} + \&c. \ldots . + \varepsilon_i,$$

and we have

$$\frac{\phi t}{\psi t} = \frac{w_1^{(\alpha)}}{t} + \frac{w_2^{(\alpha)}}{t^2} + \ldots = \Sigma \frac{w_n^{(\alpha)}}{t^n},$$

If we extend the meaning of the symbol W in some measure, by putting

$$W_i^{(\alpha)} = w_i^{(\alpha)} + \varepsilon_1 w^{(\alpha)}{}_{i-1} + \varepsilon_2 w^{(\alpha)}{}_{i-2} + \ldots + \varepsilon_{i-1} w_1^{(\alpha)},$$

and generally

$$W^{(\alpha)}{}_{i-k} = w^{(\alpha)}{}_{i-k} + \varepsilon_1 w^{(\alpha)}{}_{i-k-1} + \&c. \ldots\ldots + \varepsilon_{i-k-1} w_1^{(\alpha)}$$

for $k = 0, 1, 2 \ldots i-1$, whence

$$W_1^{(\alpha)} = w_1^{(\alpha)},$$

the numerator ϕt will be

$$\phi t = W_i^{(\alpha)} + W^{(\alpha)}{}_{i-1} t + W^{(\alpha)}{}_{i-2} t^2 + \ldots\ldots + W_1^{(\alpha)} t^{i-1}.$$

By resolving $\dfrac{\phi t}{\psi t}$ into simple fractions, we obtain an independent expression for $w_n^{(\alpha)}$. If the roots of ψt are all different and be denoted by $t_1, t_2 \ldots t_i$, we find

$$\frac{\phi t}{\psi t} = \frac{\alpha_1}{t - t_1} + \frac{\alpha_2}{t - t_2} + \ldots\ldots + \frac{\alpha_i}{t - t_i},$$

hence $$w_n^{(\alpha)} = \alpha_1 t_1^{n-1} + \alpha_2 t_2^{n-1} + \ldots\ldots + \alpha_i t_i^{n-1}.$$

The determination of the quantities ε depended upon the equations

$$W^{(\alpha)}{}_{i+1} = 0,$$

that is

$$\begin{aligned}
&w^{(1)}{}_{i+1} + \varepsilon_1 w_i^{(1)} + \varepsilon_2 w^{(1)}{}_{i-1} + \ldots + \varepsilon_i w_1^{(1)} = 0,\\
&w^{(2)}{}_{i+1} + \varepsilon_1 w_i^{(2)} + \ldots\ldots\ldots + \varepsilon_i w_1^{(2)} = 0,\\
&\quad\vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots\\
&w^{(i)}{}_{i+1} + \varepsilon_1 w_i^{(i)} + \ldots\ldots\ldots + \varepsilon_i w_1^{(i)} = 0;
\end{aligned}$$

hence there exists a definite system of values in all cases, unless the determinant $\Delta_i = \Sigma \pm w_1^1 w_2^2 w_3^3 \ldots w_i^i$, that forms the denominator of each ε, is $= 0$. But in this case the following i equations may always be satisfied:

$$\left.\begin{aligned}
&\Delta_{i-1} w_i^{(1)} + \varepsilon_1 w^{(1)}{}_{i-1} + \ldots\ldots + \varepsilon_{i-1} w_1^{(1)} = 0\\
&\Delta_{i-1} w_i^{(2)} + \varepsilon_1 w^{(2)}{}_{i-1} + \ldots\ldots + \varepsilon_{i-1} w_1^{(2)} = 0\\
&\quad\vdots \qquad\qquad\quad \vdots \qquad\qquad\qquad\quad \vdots\\
&\Delta_{i-1} w_i^{(i-1)} + \varepsilon_1 w^{(i-1)}{}_{i-1} + \ldots\ldots + \varepsilon_{i-1} w_1^{(i-1)} = 0\\
&(\Delta_{i-1} w_i^{(i)} + \varepsilon_1 w^{(i)}{}_{i-1} + \ldots\ldots + \varepsilon_{i-1} w_1^{(i)} = 0)
\end{aligned}\right\} \ldots\ldots (C),$$

therefore one at least must be a consequence of all the rest; let it be the last, which on this account has been put in brackets. The unknown quantities will now all get the

denominator

$$\Delta_{i-1} = \Sigma \pm w_1^{(1)} w_2^{(2)} w_3^{(3)} \ldots . w^{(i-1)}_{i-1},$$

which in the last system is put as a factor, so that the quantities $\varepsilon_1, \varepsilon_2 \ldots \varepsilon_{i-1}$ now denote only the corresponding numerators. Now if Δ_{i-1} is not = 0, the equations $W_n^{(\alpha)} = 0$ commence to hold from $n = i$, and the common scale of the recurring series will contain one term less. But if $\Delta_{i-1} = 0$, another of the equations (C) (let it be the last but one) is a consequence of the remaining ones, and the scale is diminished by one more term. Continuing in this manner, we suppress successively all those equations of the system (C) that are consequences of the remaining ones. Now if all the Δ's from $p = 1$ to $p = i$ were = 0, all the equations (C) would be identical, *i.e.* every $w_i^{(\alpha)}$ would = 0; that is the Knight would have no possible move at starting. This case may be excluded.

It follows that, if it should be impossible to satisfy the equations $w^{(\alpha)}_{i+1} = 0$, because Δ_i is = 0, it will always be possible to satisfy a similar system of equations $w^{(\alpha)}_{j+1} = 0$ for $\alpha = 1, 2 \ldots i$, where j is $< i$; so that we again conclude that all the $w_n^{(\alpha)}$ for different n's form recurring series, one for each value of α, and that all these series have the same scale of i terms at most.

In applications it is easy to distinguish those squares, to which, on account of the symmetry of their position, correspond equal values of w for the same values of n; and if they be denoted by the same index, we avoid the unnecessary accumulation of unknown quantities and the consequent necessity of suppressing identical equations. But when by the exclusion of some squares the symmetry is broken, these simplifications, of which I shall give an example, cease.

On a square board containing 25 squares, the Knight starts from the central field, which he is not allowed to occupy again; in how many ways can he reach each of the other squares, which he may occupy several times, in n moves?

4	3	2	3	4
3	2	1	2	3
2	1	0	1	2
3	2	1	2	3
4	3	2	3	4

The central square (0) being excluded, the remaining 24 may be arranged into four groups, marked by the figs. 1, 2, 3, 4. All squares of the same group have for any value of n the same w's: thus there are only 4 unknown quantities,

$$w_n^{(1)}, \quad w_n^{(2)}, \quad w_n^{(3)}, \quad w_n^{(4)}.$$

The diagram leads at once to the equations

$$w^{(1)}_{n+1} = 4w_n^{(2)} + 2w_n^{(4)},$$
$$w^{(2)}_{n+1} = 2w_n^{(1)} + 2w_n^{(3)},$$
$$w^{(3)}_{n+1} = 2w_n^{(2)},$$
$$w^{(4)}_{n+1} = 2w_n^{(1)}.$$

If now the quantities $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ are deduced from

$$w_5^{(\alpha)} + \varepsilon_1 w_4^{(\alpha)} + \varepsilon_2 w_3^{(\alpha)} + \varepsilon_3 w_2^{(\alpha)} + \varepsilon_4 w_1^{(\alpha)} = 0,$$

where α takes the values 1, 2, 3, 4, and the required values of w are determined from the preceding equations by means of the initial values

$$w_1^1 = 0, \quad w_1^2 = 0, \quad w_1^3 = 1, \quad w_1^4 = 0,$$

we find $\varepsilon_1 = 0, \quad \varepsilon_2 = -16, \quad \varepsilon_3 = 0, \quad \varepsilon_4 = 16;$

therefore $\psi t = t^4 - 16t^2 + 16.$

Hence we find ϕt for all four cases, as the following table shews:

$$\frac{8t}{\psi t} = \Sigma \frac{w_n^{(1)}}{t^n}; \quad \frac{2t^2-8}{\psi t} = \Sigma \frac{w_n^{(2)}}{t^n}; \quad \frac{t^3-12t}{\psi t} = \Sigma \frac{w_n^{(3)}}{t^n}; \quad \frac{16}{\psi t} = \Sigma \frac{w_n^{(4)}}{t^n}.$$

Let $\psi(t) = (t^2 - g)(t^2 - h)$, where $g = 8 + 4\sqrt{3}$, $h = 8 - 4\sqrt{3}$,

then

$$w^{(1)}_{2n+1} = \frac{g^n - h^n}{\sqrt{3}}, \qquad w^{(1)}_{2n+2} = 0,$$

$$w^{(2)}_{2n+1} = 0, \qquad w^{(2)}_{2n+2} = \frac{(\sqrt{3}+1)g^n + (\sqrt{3}-1)h^n}{\sqrt{3}},$$

$$w^{(3)}_{2n+1} = \frac{(\sqrt{3}-1)g^n + (\sqrt{3}+1)h^n}{2\sqrt{3}}, \qquad w^{(3)}_{2n+2} = 0,$$

$$w^{(4)}_{2n+1} = 0, \qquad w^{(4)}_{2n+2} = 2\frac{(g^n - h^n)}{\sqrt{3}}.$$

We are now prepared to attempt the answer of the principal question: In how many different ways can the Knight, by 63 moves, pass from a given square (a, b) or A, to another given square (a', b') or B (which of course must be of different name to that of A), without leaving any of the other squares unoccupied? Availing ourselves of the means indicated, an answer is possible, barring the really immense length of the calculation.

I call A and B the terminal squares, the rest intermediate squares, and denote them in an arbitrary order with 1, 2...62.

I make besides the condition that A is not occupied again, nor B, before the 63rd move. This condition is always to be understood where not directly expressed.

SOLUTION: Calculate the number of all the different ways of passing from A to B in 63 moves; let it be C.

Calculate the number of all such different ways of passing from A to B in 63 moves in which the intermediate square (1) is not touched; let it be C_1.

Similarly, calculate the corresponding number, when the square (2) is not touched; let it be C_2. Let C_3, $C_4 \ldots C_{62}$ have analogous meaning, and let them be calculated.

Again, let us exclude simultaneously any two intermediate squares α and β, and let $C_{\alpha\beta}$ be the number of different ways in which it is possible under this restriction to pass in 63 moves from A to B. This, applied to the combination of any two intermediate squares, will furnish $61 \times 31 = 1891$ numbers $C_{1,2}$, $C_{1,3} \ldots C_{61,62}$.

Again, let three intermediate squares be excluded at once, which will give $61.31.20 = 37820$ numbers $C_{1,2,3}$ &c. up to $C_{60,61,62}$.

Continuing thus, let successively all combinations of intermediate squares, taken 4, 5 and up to 62 together, be excluded, which will give still larger series of numbers, till we arrive at $C_{1,2,3 \ldots 60,61,62}$, which is $= 0$, as many preceding ones too. Then the required number N of all possible different ways of passing in 63 moves from A to B, without excluding and consequently also without repeatedly occupying the same square, will have the following value:

$$\begin{aligned} N = C &- \{C_1 + C_2 + C_3 + \ldots\ldots\ldots + C_{62}\} \\ &+ \{C_{1,2} + C_{1,3} + C_{2,3} + \ldots\ldots\ldots + C_{61,62}\} \\ &- \{C_{1,2,3} + \ldots\ldots\ldots\ldots\ldots + C_{60,61,62}\} \\ &+ \{C_{1,2,3,4} + \ldots\ldots\ldots\ldots\ldots + C_{59,60,61,62}\} \\ &+ \text{ &c.} \qquad\qquad\qquad\qquad \text{ &c.} \\ &+ C_{1,2,3 \ldots 61,62}. \end{aligned}$$

To prove this, let us select arbitrarily a certain number of intermediate squares, and let us see how often the number (z) of circuits in which these and only these squares remain unoccupied, enters the last formula. In C, the number of all possible circuits, it enters once; in $(C_1 + C_2 + \ldots + C_{62})$ r times, if r is the number of the squares set apart. For if the indices of these squares are 1, 2, 3 ... r, the number z occurs once in each of the numbers C_1, $C_2 \ldots C_r$, and not at all in any

of the remaining numbers of this class. In each of the numbers $C_{1,2} \ldots C_{r-1,r}$, which contain the binary combinations of the indices 1, 2 ... r, the number z enters twice, hence in $\{C_{1,2} \ldots + C_{61,62}\}$ it enters $\frac{r.r-1}{1.2}$ times; for the remaining numbers of this series do not contain it. Continuing in this manner we find that z enters in the expression for N

$$1 - r + \frac{r.r-1}{1.2} - \&c. = (1-1)^r = 0 \text{ times};$$

hence this formula gives alone the number of transitions from A to B, in which no square is excluded; and this number enters it but once. Q. E. D.

ON RECIPROCAL METHODS IN THE DIFFERENTIAL CALCULUS.

By George Boole, LL.D.

Introduction.

The design of the following paper is to establish a method in the Differential Calculus applicable to the solution of a class of problems reciprocal to those which, under the head of Envelopes of Maxima and Minima, &c., already occupy so important a place in its applications. In the direct problems referred to, one general character prevails. An equation is given involving two sets of quantities. One of these sets is made to vary, either in a perfectly unrestricted manner, or in subjection to certain stated conditions; and the object of investigation is the most general relation connecting the other set of quantities in conformity with the conditions above described. The conception of an inverse problem will now readily present itself. Given the original equation connecting two sets of quantities, required the conditions necessary to connect one set among themselves, in order that, however, in subjection to those conditions, that set may vary, the other set of quantities may, among themselves, be subject to a given relation. The complete solution of this problem is contained in the present essay, but the results which it involves admit of being stated, and it may be desirable here to state them, in theorems of great simplicity. If, in analogy with admitted language, we term that set of quantities which, by their variation in sub-

jection to given conditions, establish a relation among the other set of quantities, *parameters*, and that other set among which the relation is established, *coordinates;* in doing which we extend, for convenience, the use of the term coordinate to subjects analogous only in their analytical relations to the coordinates of geometry: then the question we have to consider will be resolvable into three cases, to the solution of which, separate but connected theorems are applicable.

CASE I. When the number of the parameters is equal to the number of the coordinates.

THEOREM. If $\phi(x_1.x_2\,..\,x_n,\ a_1.a_2\,..\,a_n) = 0$ be the given equation, in which $a_1.a_2\,..\,a_n$ are parameters subject to any relation $\psi(a_1.a_2\,..\,a_n) = 0$, in virtue of which the coordinates $x_1.x_2\,..\,x_n$ are subject to a relation $\chi(x_1.x_2\,..\,x_n) = 0$, then, *reciprocally*, if we regard $x_1.x_2\,..\,x_n$ as *parameters* subject to the relation $\chi(x_1.x_2\,..\,x_n) = 0$, and $a_1.a_2\,..\,a_n$ as coordinates, the relation among the latter will be $\psi(a_1.a_2\,..\,a_n) = 0$.

In other words, *the parameters with their relations are mutually interchangeable with the coordinates and their relations.*

CASE II. When the number of the parameters exceeds that of the coordinates.

THEOREM. Let $\phi(x_1.x_2\,..\,x_m,\ a_1.a_2\,..\,a_n) = 0$ be an equation, in which $a_1.a_2\,..\,a_n$ are parameters and $x_1.x_2\,..\,x_m$ coordinates, and let $n > m$. Then the most general relations among the parameters, in virtue of which the coordinates may be made subject to a relation $\chi(x_1.x_2\,..\,x_m) = 0$, will be a system of $n - m + 1$ equations between $a_1.a_2\,..\,a_n$, whereof $n - m$ are perfectly arbitrary, and the remaining and only essential one is the same which we should obtain by regarding in the primitive equation $a_1.a_2\,..\,a_n$ as *coordinates*, and $x_1.x_2\,..\,x_m$ as parameters, subject to the given relation $\chi(x_1.x_2\,..\,x_m) = 0$.

CASE III. When the number of parameters is less than that of coordinates.

Let $\phi(x_1.x_2\,..\,x_m,\ a_1.a_2\,..\,a_n) = 0$ be the given equation, and let n, the number of the parameters, be less than m. It is not then in general possible, by establishing any relation among $a_1.a_2\,..\,a_n$, to necessitate a proposed relation $\chi(x_1.x_2\,..\,x_m) = 0$, *and no other*, among the coordinates $x_1.x_2\,..\,x_m$. But if, as before, we regard $x_1.x_2\,..\,x_m$ as parameters, the resulting relation among $a_1.a_2\,..\,a_n$ will be such as to necessitate a system of equations among $x_1.x_2\,..\,x_m$, in which the proposed

equation will be involved. This case is exemplified whenever we require to cause a surface of a given species or character to move in perpetual contact with another fixed surface, through the variation of the arbitrary contents in its equation. The previous cases relate to problems such as the determination of the conditions among the arbitrary contents, in order that a given surface may have a given envelope. All these cases are also related to the inverse problem of maxima and minima.

Of the theorems above stated the first appears to be the most important, and it really constitutes the basis upon which the others rest. It also involves as a particular consequence a theory of reciprocal polars, of which the received theory appears to be a special form.

Let $\phi(x, y, z, x', y', z') = 0$ be the equation of a surface in which x, y, z represent coordinates and x', y', z' parameters, the latter being subject to a condition $\psi(x', y', z') = 0$; and let the equation $\chi(x, y, z) = 0$ be thence deduced as the equation of the envelope of the surface given. Reciprocally, if in the primitive equation we regard x, y, z as parameters subject to the condition $\chi(x, y, z) = 0$, and x', y', z' as coordinates, then will $\psi(x', y', z') = 0$ express the envelope of the surface now represented by the primitive equation.

Suppose, then, that in that equation x, y, z and x', y', z' enter *symmetrically*, so that on changing x, y, z into x', y', z' respectively, and *vice versa*, the equation remains unchanged: also, any point x, y, z being given, let the above equation represent the *polar surface* of that point, x', y', z' being coordinates and x, y, z parameters; whence, by virtue of the symmetry, any point x', y', z' being given, the same equation, with x, y, z as coordinates, will represent the *polar surface* of that point. Now, by the theorem, if the surface $\chi(x, y, z) = 0$ is generated by the intersections of the polar surfaces of all the points contained on the surface $\psi(x', y', z') = 0$, then *reciprocally* the surface $\psi(x', y', z') = 0$ will be generated by the successive intersections of the polar surfaces of the points found on the surface $\chi(x, y, z) = 0$. In the received theory, the polar surface of a point, or pole as it is there termed, is a plane whose equation is of the form $xx' + yy' + zz' = 1$, or, in the most general case, of the form $axx' + byy' + czz' = 1$. The sole requisite condition of symmetry is therefore answered. But there is in the nature of things no necessity for the restriction of the equation of the polar of a point to the first degree. There is no reason why, in Geometry of Two Dimensions, the pole of a point

should not be an ellipse, or in Geometry of Three Dimensions an ellipsoid. I am not aware that the theory of reciprocal polars has been viewed in this universal manner before,—a manner which I think also to be more satisfactory than that which has previously been accepted, as well as more general.*

In the following demonstrations I have sometimes, when it appeared possible to avoid complexity without incurring the danger of misapprehension, substituted the three variables x, y, z for the more general system. Some of the conclusions stated in this paper may, I believe, be obtained by shorter methods than the one which is actually employed; certainly they may if they are regarded solely in their geometrical aspect and limited in expression accordingly. But if considered as purely quantitative relations, the special views of geometry are no longer applicable, and some new principle of investigation is needed. That which I have employed appears to me to be new and in some respects important. It is, that the equation or equations among the parameters are transformations of those which connect the coordinates, or speaking more generally, that the set of quantities among which relation is given, and the set among which relation is sought, are so connected, that the one relation or set of relations is but a transformation of the other. This principle being established, the known doctrines relative to the transformation of functions become applicable, and the subject is placed in that point of view which is apparently the most general.

Reciprocal Methods in the Differential Calculus.

THEOREM I. If two sets of quantities $x_1, x_2 \ldots x_n$ and $a_1, a_2 \ldots a_n$, equal in number, are connected by any relation or relations $u_1 = 0, u_2 = 0 \ldots u_r = 0$, and if the former set, further varying in subjection to a condition among themselves, $X = 0$, establish among the set $a_1, a_2 \ldots a_n$ a condition $A = 0$; then, conversely, the set $a_1, a_2 \ldots a_n$, varying in subjection to the condition $A = 0$, will establish among the quantities $x_1, x_2 \ldots x_n$, with which they are connected, the relation $X = 0$.

* Mr. Cayley informs me that the theory of reciprocal polars has actually been presented in this form by Druckenmüller (*Crelle*, tom. XXVI. p. 1). The manuscript of the present memoir has been lying by me for about a year and a half, and I have during that period had but little opportunity of consulting foreign journals. I think it proper, however, to retain the above brief notice of the theory of polars, as possibly new to some of the readers of the *Journal*, and at the same time affording an interesting illustration of one of the general theorems of the present memoir.

The equations in which $x_1, x_2 \ldots x_n$ enter are

$$X = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$u_1 = 0, \quad u_2 = 0 \ldots u_r = 0 \ldots\ldots\ldots (2).$$

By affecting the latter set of equations with indeterminate multipliers, and adding to the former, we get

$$X + \lambda_1 u_1 + \lambda_2 u_2 \ldots + \lambda_r u_r = 0,$$

which we will represent by $U = 0$, and its differentiation will furnish the system of equations

$$\frac{dU}{dx_1} = 0, \quad \frac{dU}{dx_2} = 0, \quad \frac{dU}{dx_n} = 0 \ldots\ldots\ldots (3).$$

From the $r + n$ equations in (2) and (3), we can determine the values of $\lambda_1, \lambda_2 \ldots \lambda_r$ and $x_1, x_2 \ldots x_n$ in terms of $a_1, a_2 \ldots a_n$, and the substitution of the values thus found for $x_1, x_2 \ldots x_n$ in the equation (1) or $X = 0$ will give a relation among the constants $a_1, a_2 \ldots a_n$ which we may represent by $A = 0$. This is that relation which is so represented in the statement of the theorem, and which we are now to shew possessed of reciprocal properties with C.

Since by the above, A is a transformation of X, we may write

$$X = A,$$

or

$$X - A = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4);$$

and, connecting this equation with the system (2) by indeterminate multipliers, we have

$$X - A + \lambda_1 u_1 + \lambda_2 u_2 \ldots + \lambda_r u_r = 0 \ldots\ldots (5),$$

or

$$U - A = 0.$$

Differentiate this equation with reference to $x_1, x_2 \ldots x_n$ and $a_1, a_2 \ldots a_n$, we get, on transposition,

$$\frac{dU}{dx_1} dx_1 + \frac{dU}{dx_2} x_2 \ldots + \frac{dU}{dx_n} dx_n$$

$$= \frac{d(A - U)}{da_1} da_1 + \frac{d(A - U)}{da_2} da_2 \ldots + \frac{d(A - U)}{da_n} da_n,$$

whence, as the number of the differentials $da_1, da_2 \ldots da_n$ is equal to the number of the differentials $dx_1, dx_2 \ldots dx_n$, the one set moreover being linear functions of the other, it appears that the two systems of equations

$$\frac{dU}{dx_1} = 0, \quad \frac{dU}{dx_2} = 0, \ldots \quad \frac{dU}{dx_n} = 0 \ldots\ldots\ldots (6),$$

$$\frac{d(A-U)}{da_1}=0,\quad \frac{d(A-U)}{da_2}=0,\ldots\quad \frac{d(A-U)}{da_n}=0\ldots(7),$$

are equivalent; *i.e.* the satisfaction of the one set of relations involves that of the other.

But the set (6) is identical with (3), and its satisfaction furnishes the means of deducing A from X in accordance with the conditions of the problem stated in the outset. Again, if in (7) we write for U its value, viz.

$$X+\lambda_1u_1+\lambda_2u_2\ldots+\lambda_ru_r,$$

and observe that X does not involve $a_1, a_2 \ldots a_n$, we get the system of equations,

$$\frac{d(A-\lambda_1u_1-\lambda_2u_2\ldots-\lambda_ru_r)}{da_1}=0,$$

$$\frac{d(A-\lambda_1u_1-\lambda_2u_2\ldots-\lambda_ru_r)}{da_2}=0,$$

&c. &c.

$$\frac{d(A-\lambda_1u_1-\lambda_2u_2\ldots-\lambda_ru_r)}{da_n}=0.$$

But these are the very equations which we should form if we sought to determine the relation among $x_1, x_2 \ldots x_n$ consequent upon the variations of $a_1, a_2 \ldots a_n$ in the most general manner in subjection to the relation $A=0$, and to the connecting system of relations $u_1=0$, $u_2=0$, $u_n=0$.

Let $P=0$ represent the relation thus formed among $x_1, x_2 \ldots x_n$. Then it appears that the total system of relations necessary to transform X into A is equivalent to the total system necessary to retransform A into P. Wherefore P is equivalent to X, and the functions X and A possess reciprocal properties. Whence the theorem is proved.

Let $\phi(x, y, z, a, b, c)=0$ represent the equation of a proposed surface, x, y, z being the coordinates and a, b, c the constants. If the latter be made to vary in subjection to any condition $\psi(a, b, c)=0$, the form of the surface will vary, and the locus of its successive intersections will in general constitute a new surface $\chi(x, y, z)=0$, which is said to be the envelope of the system of surfaces defined by the equation $\phi(x, y, z, a, b, c)=0$ under the proposed condition. The constants a, b, c, whether supposed to vary in the manner above described, or to be free from all restriction, are termed the *parameters*.

To apply to this case the general theorem, it is only necessary to observe that the system of equations (2) here is reducible to the single equation $\phi(x, y, z, a, b, c) = 0$, the equation X will be represented by $\chi(x, y, z) = 0$, and the equation $A = 0$ by $\psi(a, b, c) = 0$. We are thus led to the following theorem.

THEOREM. If $\chi(x, y, z) = 0$ is the equation of the envelope of surface whose equation is $\phi(x, y, z, a, b, c) = 0$, wherein a, b, c are parameters varying in subjection to the sole condition $\psi(a, b, c) = 0$, then is $\psi(a, b, c) = 0$ the equation of the envelope of the surface $\phi(x, y, z, a, b, c) = 0$, when x, y, z are regarded as parameters, subject to the condition $\chi(x, y, z) = 0$, and a, b, c are coordinates.

In other words, if the equation of a curve or surface involve an equal number of coordinates and parameters, the latter being subject to an equation of condition, then, on changing in the primitive equation the parameters into coordinates and the coordinates into parameters, the equation of condition and the equation of the envelope will mutually, but merely, change their places.

Hence to determine the condition under which a given curve or surface in which the number of parameters is equal to that of coordinates shall have a given envelope, it is only necessary to seek the envelope of that surface, regarding the coordinates as parameters and the equation of the envelope as the equation of condition among those parameters.

We might also arrive at the result required, by considering that a proposed curve or surface, which by the variation of its parameters is made to produce a given envelope, is always in contact with that envelope, and therefore has always along the line of contact the same tangent planes as the envelope. I am not sure that we should not by this method arrive somewhat more readily at the rule for determining the required relations among the parameters, but we should, in following this track, in a great measure lose sight of that principle of *reciprocity* which the more purely analytical treatment of the subject sets in so clear a light before us.

Examples.

Ex. 1. Let it be required to determine the condition under which the straight line whose equation is

$$\frac{x}{m} + \frac{y}{n} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

will have for its envelope an ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots\ldots\ldots\ldots\ldots (9).$$

Here the equation of the straight line involves two coordinates x, y, and two parameters m, n, and it is the relation connecting the two latter which is sought.

Regarding then x and y as *parameters*, and considering the equation of the envelope (12) as an equation between those parameters, we get in the ordinary way

$$\frac{1}{m}\,dx + \frac{1}{n}\,dy = 0,$$

$$\frac{x}{a^2}\,dx + \frac{y}{b^2}\,dy = 0,$$

whence

$$\frac{y}{mb^2} - \frac{x}{na^2} = 0 \ldots\ldots\ldots\ldots (10).$$

Eliminating x and y between (8), (9), and (10), we have

$$\frac{a^2}{m^2} + \frac{b^2}{n^2} = 1 \ldots\ldots\ldots\ldots (11).$$

To verify this it is only necessary to seek the envelope of (8), regarding m and n as parameters subject to the condition (11). The result is (9).

Ex. 2. Under what conditions will the plane whose equation is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \ldots\ldots\ldots\ldots (12),$$

have for its envelope the surface whose equation is

$$xyz = m^3 \ldots\ldots\ldots\ldots\ldots (13).$$

We have, in obedience to the rule,

$$\frac{dx}{a} + \frac{dy}{b} + \frac{dz}{c} = 0,$$

$$yz\,dx + zx\,dy + xy\,dz = 0;$$

whence, on multiplying the first equation by λ, adding to it the second, and equating to 0 the coefficients of the differentials, we get

$$\frac{\lambda}{a} + yz = 0, \quad \frac{\lambda}{b} + xz = 0, \quad \frac{\lambda}{c} + xy = 0,$$

whence $$ayz = bxz = cxy.$$

Eliminating x, y, and z between these equations and the two primitives, we find

$$abc = \frac{m^3}{27} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

for the condition sought, a result easily verified.

Ex. 3. Required the conditions under which the circle whose equation is $(x-a)^2 + (y-b)^2 = r^2$ shall have for its envelope the circle whose equation is $x^2 + y^2 = 1$, a and b being the variable parameters.

We have $$(x-a)^2 + (y-b)^2 = r^2 \quad \ldots\ldots\ldots \quad (15),$$

$$x^2 + y^2 = 1 \quad \ldots\ldots\ldots\ldots\ldots \quad (16),$$

whence, regarding x and y as parameters,

$$\left.\begin{aligned}(x-a)\,dx + (y-b)\,dy &= 0\\ x\,dx + y\,dy &= 0\end{aligned}\right\};$$

and, eliminating dx and dy,

$$(x-a)\,y - (y-b)\,x = 0,$$

or $$bx = ay \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17).$$

From (16) and (17), we have

$$x = \frac{a}{\sqrt{(a^2+b^2)}}, \qquad y = \frac{b}{\sqrt{(a^2+b^2)}};$$

and substituting these values in (15), and reducing, we find

$$a^2 + b^2 = (1 \pm r)^2 \quad \ldots\ldots\ldots\ldots \quad (18)$$

as the condition sought. This implies that the centre of the primitive circle whose radius is r must move along the circumference of another circle whose radius is $1 + r$ or $1 \sim r$, in order to have for its envelope the proposed circle whose radius is unity. This is obviously true. It will be remarked that the proposed envelope, the circle whose radius is unity, is but part of the entire envelope which would be produced by the above motion, and which would in fact consist of three concentric circles whose respective radii are $1 - 2r$, 1, and $1 + 2r$.

As the sole condition recognised in the theorem which the above examples serve to illustrate is that the number of the parameters shall be equal to the number of the coordinates, and as in the last example there occur three

constants a, b, r, and two coordinates x, y; it is apparent that instead of employing a and b as parameters, we might have employed either of the other pairs a, r and b, r, and that the same solution will apply to all the three cases. Thus it appears that the circle represented by (18) will produce the envelope (19) when any two of the constants a, b, r are made to vary in subjection to the condition (18). We have considered the case in which a and b vary. Let us next suppose that a and r are the varying parameters, and merely for convenience let us suppose that r is greater than unity. Then, writing (17) and (18) in the forms

$$(x-a)^2+(y-b)^2-r^2=0 \ldots\ldots\ldots (19),$$

$$a^2+b^2-(r\pm 1)^2=0 \ldots\ldots\ldots (20);$$

and, differentiating with reference to a and r, we get

$$-(x-a)\,da-rdr=0,$$

$$adu-(r\pm 1)\,dr=0;$$

and, eliminating the differentials,

$$(x-a)(r\pm 1)+ar=0.$$

If now between this equation and the two primitives (19) and (20) we eliminate a and b, we obtain a result equivalent to the two following equations, viz.

$$x^2+y^2=1,$$

$$x^2+(y-2b)^2=1.$$

The former of these equations agrees with (16), the latter of them represents another circle having the same radius and having its centre situated in the axis of y at a distance $2b$ from the origin. Now it is obvious that the variable circle (19) cannot, under the circumstances supposed, viz. b *not* varying, generate the first of the above circles, which was the one required to be generated, without also generating the second. The following is the true theory of the case.

Whenever we apply our theorem to determine the condition under which a given curve or surface shall by its successive intersections generate a given envelope, we ascertain that condition in the form of an equation among the constants of the given curve or surface. Provided that this equation is satisfied, it is indifferent which two (in the curve) or which three (in the surface) of the said constants we suppose to vary. In *every* case we obtain the required envelope, but in *any* case we may also obtain additional

branches, and these will differ as different sets of constants are supposed to vary.

A part of these conclusions is indeed sufficiently obvious. We cannot cause a sphere to move so as to produce a given sphere for its envelope, without its also generating some other surface as another envelope. Our analysis gives the common condition for determining both these.

[*To be continued.*]

ON THE SINGULARITIES OF SURFACES.

By Arthur Cayley.

In the following paper, for symmetry of nomenclature and in order to avoid ambiguities, I shall, with reference to plane curves and in various phrases and compound words, use the term "node" as synonymous with double point, and the term "spinode" as synonymous with cusp. I shall, besides, have occasion to consider the several singularities which I call the "flecnode," the "oscnode," the "fleflecnode," and the "tacnode:" the flecnode is a double point which is a point of inflexion on one of the branches through it; the oscnode is a double point which is a point of osculation on one of the branches through it; the fleflecnode is a double point which is a point of inflexion on each of the branches through it; and the tacnode is a double point where two branches touch. And it may be proper to remark here, that a tacnode may be considered as a point resulting from the coincidence and amalgamation of two double points (and therefore equivalent to twelve points of inflexion); or, in a different point of view, as a point uniting the characters of a spinode and a flecnode. I wish to call to mind here the definition of conjugate tangent lines of a surface, viz. that a tangent to the curve of contact of the surface with any circumscribed developable and the corresponding generating line of the developable, are conjugate tangents of the surface.

Suppose, now, that an absolutely arbitrary surface of any order be intersected by a plane: the curve of intersection has not in general any singularities other than such as occur in a perfectly arbitrary curve of the same order; but as a plane can be made to satisfy one, two, or three conditions, the curve may be made to acquire singularities which do not occur in such absolutely arbitrary curve.

Let a single condition only be imposed on the plane. We may suppose that the curve of intersection has a node; the plane is then a tangent plane and the node is the point of contact—of course any point on the surface may be taken for the node. We may if we please use the term "nodes of a surface," "node-planes of a surface," as synonymous with the points and tangent planes of a surface. And it will be convenient also to use the word node-tangents to denote the tangents to the curve of intersection at the node; it may be noticed here that the node-tangents are conjugate tangents of the surface.

Next let two conditions be imposed upon the plane: there are three distinct cases to be considered.

First, the curve of intersection may have a flecnode. The plane (which is of course still a tangent plane at the flecnode) may be termed a flecnode-plane; the flecnodes are singular points on the surface lying on a curve which may be termed the "flecnode-curve,"* and the flecnode-planes give rise to a developable which may be termed the flecnode-develope. The "flecnode-tangents" are the tangents to the curve of intersection at the flecnode; the tangent to the inflected branch may be termed the "singular flecnode-tangent," and the tangent to the other branch the "ordinary flecnode-tangent."

Secondly, the curve of intersection may have a spinode. The plane (which is of course still a tangent plane at the spinode) may be termed a spinode-plane; the spinodes are singular points on the surface lying on a curve which may be termed the "spinode-curve."† And the spinode-planes give rise to a developable which may be termed the "spinode-develope." Also the "spinode-tangent" is the tangent to the curve of intersection at the spinode.

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* The flecnode-curve, defined as the locus of the points through which can be drawn a line meeting the surface in four consecutive points, was, so far as I am aware, first noticed in Mr. Salmon's paper On the Triple Tangent Planes of a Surface of the Third Order (*Journal*, tom. IV. p. 252), where Mr. Salmon, among other things, shews that the order of the surface being n, the curve in question is the intersection of the surface with a surface of the order $11n-24$.

† The notion of a spinode, considered as the point where the indicatrix is a parabola (on which account the spinode has been termed a parabolic point) may be found in Dupin's *Developpements de Geometrie:* the most important step in the theory of these points is contained in Hesse's memoir "Ueber die Wendepuncte der Curven dritter Ordnung," (*Crelle*, tom. XXVIII. p. 97), where it is shewn that the spinode-curve is the curve of intersection of the surface supposed as before of the order n, with a certain surface of the order $4(n-2)$. See also Mr. Salmon's memoir "On the Condition that a Plane should touch a Surface along a Curve Line," (*Journal*, tom. III. p. 44.)

Thirdly, the curve of intersection may have two nodes, or what may be termed a "node-couple." The plane (which is a tangent plane at each of the nodes and therefore a double tangent plane) may be also termed a "node-couple-plane." The node-couples are pairs of singular points on the surface lying in a curve which may be termed the "node-couple-curve," and the node-couple-planes give rise to a developable which may be termed the "node-couple-develope." The tangents to the curve of intersection at the two nodes of a node-couple might, if the term were required, be termed the "node-couple-tangents." Also one of the nodes of a node-couple may be termed a "node-with-node," and the tangents to the curve of intersection at such point will be the "node-with-node-tangents."

It is hardly necessary to remark that the flecnode-curve is *not* the edge of regression of the flecnode-develope, and the like remark applies *m.m.* to the spinode-curve and the node-couple curve.

Finally, let three conditions be imposed upon the plane: there are six distinct cases to be considered, in each of which we have no longer curves and developes, but only singular points and singular tangent planes determinate in number.

First, the curve of intersection may have an oscnode. The plane (which continues a tangent plane at the oscnode) is an "oscnode-plane." The "oscnode-tangents" are the tangents to the curve of intersection at the oscnode; the tangent to the osculating branch is the "singular oscnode-tangent;" and the tangent to the other branch the "ordinary oscnode-tangent."

Secondly, the curve of intersection may have a fleflecnode. The plane (which continues a tangent plane at the fleflecnode) is a "fleflecnode-plane." The "fleflecnode-tangents" are the tangents to the curve of intersection at the fleflecnode.

Thirdly, the curve of intersection may have a tacnode. The plane (which continues a tangent plane at the tacnode) is a "tacnode-plane." The "tacnode-tangent" is the tangent to the curve of intersection at the tacnode.

Fourthly, the curve of intersection may have a node and a flecnode, or what may be termed a node-and-flecnode. The plane (which is a tangent plane at the node and also at the flecnode, where it is obviously a flecnode-plane) is a "node-and-flecnode-plane." The "node-and-flecnode-tangents," if the term were required, would be the tangents

to the curve of intersection at the node and at the flecnode of the node-and-flecnode. The node of the node-and-flecnode may be distinguished as the node-with-flecnode, and the flecnode as the flecnode-with-node, and we have thus the terms "node-with-flecnode-tangents," "flecnode-with-node-tangents," "singular flecnode-with-node-tangent," and "ordinary flecnode-with-node-tangent."

Fifthly, the curve of intersection may have a node and also a spinode, or what may be termed a "node-and-spinode." The plane (which is a tangent plane at the node and also a tangent plane at the spinode, where it is obviously a spinode-plane) is a "node-and-spinode-plane." The node-and-spinode-tangents, if the term were required, would be the tangents at the node and the tangent at the spinode of the node-and-spinode to the curve of intersection. The node of the node-and-spinode may be distinguished as the "node-with-spinode," and the spinode as the "spinode-with-node," and we have thus the terms "node-with-spinode-tangent," "spinode-with-node-tangent."

Sixthly, the curve of intersection may have three nodes, or what may be termed a "node-triplet." The plane (which is a triple tangent plane touching the surface at each of the nodes) is a "node-triplet-plane." The "node-triplet-tangents," if the term were required, would be the tangents to the curve of intersection at the nodes of the node-triplet. Each node of the node-triplet may be distinguished as a "node-with-node-couple," and the tangents to the curve of intersection at such nodes are "node-with-node-couple-tangents." The terms "node-couple-with-node," "node-couple-with-node tangent", might be made use of if necessary.

It should be remarked that the oscnodes lie on the flecnode-curve, as do also the fleflecnodes; these latter points are real double points of the flecnode-curve. The tacnodes are points of intersection and (what will appear in the sequel) points of contact of the flecnode-curve, the spinode-curve, and the node-couple-curve. The spinode-with-nodes are points of intersection of the spinode-curve and node-couple-curve, and the flecnode-with-nodes are points of intersection of the flecnode-curve and node-couple curve; the node-with-node-couples are real double points (entering in triplets) of the node-couple-curve.

Consider for a moment an arbitrary curve on the surface, the locus of the node-tangents at the different points of this curve is in general a skew surface, which may however, in cases to be presently considered, degenerate in different ways.

Reverting now to the flecnode-curve, it may be shewn that the singular flecnode-tangent coincides with the tangent of the flecnode-curve. For consider on a surface two consecutive points such that the line joining them meets the surface in two points consecutive to the first-mentioned two points. The line meets the surface in four consecutive points, it is therefore a singular flecnode-tangent; *each* of the first-mentioned two points must be on the flecnode-curve, or the singular flecnode-tangent touches the flecnode-curve. The two flecnode-tangents are by a preceding observation conjugate tangents. It follows that the skew surface, locus of the flecnode-tangents, breaks up into two surfaces, each of which is a developable, viz. the locus of the singular flecnode-tangents is the developable having the flecnode-curve for its edge of regression, and the locus of the ordinary flecnode-tangents is the flecnode-develope. Of course at the tacnode, the tacnode-tangent touches the flecnode-curve.

Passing next to the spinode-curve, the spinode-plane and the tangent-plane at a consecutive point along the spinode-tangent are identical,* or their line of intersection is indeterminate. The spinode tangent is therefore the conjugate tangent to *any* other tangent line at the spinode, and therefore to the tangent to the spinode-curve. It follows that the surface locus of the spinode-tangents degenerates into a developable surface twice repeated, viz. the spinode-develope. Consider the tacnode as two coincident nodes; each of these nodes, by virtue of its constituting, in conjunction with the other, a tacnode, is on the spinode-curve; or, in other words, the tacnode-tangent touches the spinode-curve, and the same reasoning proves that it touches the node-couple-curve. It has already been seen that the tacnode-tangent touches the flecnode-curve; consequently the tacnode is a point not of simple intersection only, but of contact of the flecnode-curve, the spinode-curve, and the node-couple-curve.

In virtue of the principle of the spinode-plane being identical with the tangent plane at a consecutive point along the spinode tangent, it appears that the tacnode-plane is a stationary plane, as well of the flecnode-develope as of the spinode-develope, and it would at first sight appear that it must be also a stationary tangent plane of the node-

* It must not be inferred that the tangent plane at such consecutive point is a spinode plane; this is obviously not the case.

couple-develope. But this is not so; the node-with-node-planes envelope, not the node-couple-develope, but the node-couple-develope twice repeated: the tacnode-plane is in a sense a stationary plane on such duplicate developable, but not in any manner on the single developable. The tacnode-plane is an ordinary tangent plane of the node-couple-develope.

Consider now a spinode-with-node, which we have seen is a point of intersection of the spinode-curve and node-couple-curve. The tangent plane at a consecutive point along the spinode-with-node-tangent, is *identical* with the spinode-with-node-plane; the curve of intersection of the tangent plane at such consecutive point has therefore a node at the node-with-spinode, or the tangent plane in question is a node-couple-plane, and the point of contact is a point on the node-couple-curve. Consequently the spinode-with-node-tangent touches the node-couple-curve, and thence also the spinode-with-node-plane is a stationary tangent plane of the node-couple-develope.

It should be remarked that no circumscribed developable can have a stationary tangent plane except the tangent planes at the points where the curve of contact meets the spinode-curve, and any one of these planes is only a stationary plane when the curve of contact touches the spinode-tangent; and that the node-couple-curve and the flecnode-curve do not intersect the spinode-curve except in the points which have been discussed.

Recapitulating, the node-couple-curve and the spinode-curve touch at the tacnodes, and intersect at the spinode-with-nodes: moreover, the tacnode-planes are stationary planes of the spinode-develope, and the spinode-with-node-planes are stationary planes of the node-couple-develope. Besides this, the two curves are touched at the tacnodes by the flecnode-curve, and the tacnode-planes are stationary planes of the flecnode-develope.

ON THE THEORY OF SKEW SURFACES.

By Arthur Cayley.

A SURFACE of the n^{th} order is a surface which is met by an indeterminate line in n points. It follows immediately that a surface of the n^{th} order is met by an indeterminate plane in a curve of the n^{th} order.

Consider a skew surface or the surface generated by a singly infinite series of lines, and let the surface be of the n^{th} order. Any plane through a generating line meets the surface in the line itself and in a curve of the $(n-1)^{th}$ order. The generating line meets this curve in $(n-1)$ points. Of these points one, viz. that adjacent to the intersection of the plane with the consecutive generating line, is a unique point; the other $(n-2)$ points form a system. Each of the $(n-1)$ points are *sub modo* points of contact of the plane with the surface, but the proper point of contact is the unique point adjacent to the intersection of the plane with the consecutive generating line. Thus every plane through a generating line is an ordinary tangent plane, the point of contact being a point on the generating line. It is not necessary for the present purpose, but I may stop for a moment to refer to the known theorems that the anharmonic ratio of any four tangent planes through the same generating line is equal to the anharmonic ratio of their points of contact, and that the locus of the normals to the surface along a generating line is a hyperbolic paraboloid. Returning to the $(n-2)$ points in which, together with the point of contact, a generating line meets the curve of intersection of the surface and a plane through the generating line, these are fixed points independent of the particular plane, and are the points in which the generating line is intersected by other generating lines. There is therefore on the surface a double curve intersected in $(n-2)$ points by each generating line of the surface—a property which, though insufficient to determine the order of this double curve, shews that the order cannot be less than $(n-2)$. (Thus for $n=4$, the above reasoning shews that the double-curve must be at least of the second order: assuming for a moment that it is in any case precisely of this order, it obviously cannot be a plane curve, and must therefore be two nonintersecting lines. This suggests at any rate the existence of a class of skew surfaces of the fourth order generated by a line which always passes through two fixed lines and by some other condition not yet ascertained; and it would appear that surfaces of the second order constitute a degenerate species belonging to the class in question).

In particular cases a generating line will be intersected by the consecutive generating line. Such a generating line touches the double curve.

Consider now a point not on the surface, the planes determined by this point and the generating lines of the

surface are the tangent planes through the point, the intersections of consecutive tangent planes are the tangent lines through the point, and the cone generated by these tangent lines or enveloped by the tangent planes is the tangent cone corresponding to the point. This cone is of the n^{th} class. For considering a line through the point, this line meets the surface in n points, *i.e.* meets n generating lines of the surface and the planes through the line and these n generating lines, are of course tangent planes to the cone; that is, n tangent planes can be drawn to the cone through a given line passing through the vertex. The cone has not in general any lines of inflexion, or, what is the same thing, stationary tangent planes. For a stationary tangent plane would imply the intersection of two consecutive generating lines of the surface. And since the number of generating lines intersected by a consecutive generating line, and therefore the number of planes through two consecutive generating lines is finite, no such plane passes through an indeterminate point. The tangent cone will have in general a certain number of double tangent planes; let this number be x. We have therefore a cone of the class n, number of double tangent planes x, number of stationary tangent planes 0. Hence, if m be the order of the cone, α the number of its double lines, and β the number of its cuspidal or stationary lines,

$$m = n(n-1) - 2x,$$

$$\beta = 3n(n-2) - 6x,$$

$$\alpha = \tfrac{1}{2}n(n-2)(n^2-9) - 2x(n^2-n-6) + 2x(x-1).$$

This is the proper tangent cone, but the cone through the double curve is *sub modo* a tangent cone, and enters as a square factor into the equation of the general tangent cone of the order $n(n-1)$. Hence, if X be the order of the double curve, and therefore of the cone through this curve,

$$m + 2X = n(n-1), \quad \text{and therefore } X = x;$$

that is, the number of double tangent planes to the tangent cone is equal to the order of the double curve. It does not appear that there is anything to determine x. And if this is so, skew surfaces of the n^{th} order may be considered as forming different families according to the order of the double curve upon them.

To complete the theory, it should be added that a plane intersects the surface in a curve of the n^{th} order having x double points but no cusps.

ON CERTAIN MULTIPLE INTEGRALS CONNECTED WITH THE THEORY OF ATTRACTIONS.

By Arthur Cayley.

It is easy to deduce from Mr. Boole's formula, given in my paper "On a Multiple Integral connected with the theory of Attractions," *Journal*, tom. II. p. 219, the equation

$$\int \frac{d\xi\, d\eta \ldots}{[(\xi-a)^2+(\eta-\beta)^2+\ldots v^2]^{\frac{1}{2}n-q}} = \frac{fg\ldots\pi^{\frac{1}{2}n}}{\theta_1^{\,n}\,\Gamma(\frac{1}{2}n-q)\,\Gamma(q+1)} \int_\varepsilon^\infty \frac{s^{q-1}(\theta_1^{\,2}-\sigma)^q\, ds}{\sqrt{\left\{\left(s+\frac{f^2}{\theta_1^{\,2}}\right)\left(s+\frac{g^2}{\theta_1^{\,2}}\right)\right\}\ldots}}$$

where n is the number of variables of the multiple integral, and the condition of the integration is

$$\frac{(\xi-a_1)^2}{f^2}+\frac{(\eta-\beta_1)^2}{g^2}+\ldots \leqq 1.$$

Also where

$$\sigma = \frac{(a-a_1)^2}{s+\frac{f^2}{\theta_1^{\,2}}}+\frac{(\beta-\beta_1)^2}{s+\frac{g^2}{\theta_1^{\,2}}}\ldots+\frac{v^2}{s},$$

and ε is the positive root of

$$\theta_1^{\,2} = \frac{(a-a_1)^2}{\varepsilon+\frac{f^2}{\theta_1^{\,2}}}+\frac{(\beta-\beta_1)^2}{\varepsilon+\frac{g^2}{\theta_1^{\,2}}}\ldots+\frac{v^2}{\varepsilon}.$$

Suppose $f=g\ldots=\theta_1$, and write $(a-a_1)^2+\ldots=k^2$, we obtain

$$\int \frac{d\xi\ldots}{[(\xi-a)^2+\ldots v^2]^{\frac{1}{2}n-q}} = \frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n-q)\,\Gamma(q+1)} \int_\varepsilon^\infty \frac{s^{q-1}(\theta_1^{\,2}-\sigma)^q\, ds}{(1+s)^{\frac{1}{2}n}},$$

the limiting condition for the multiple integral being

$$(\xi-a_1)^2+\ldots \leqq \theta_1^{\,2},$$

and the function σ and limit ε being given by

$$\sigma = \frac{k^2}{1+s}+\frac{v^2}{s}, \quad \theta_1^{\,2} = \frac{k^2}{1+\varepsilon}+\frac{v^2}{\varepsilon},$$

ε denoting, as before, the positive root. Observing that the quantity under the integral sign on the second side vanishes for $s=\varepsilon$, there is no difficulty in deducing, by a differentiation with respect to θ_1, the formula

$$\int \frac{d\Sigma}{[(\xi-a)^2\ldots+v^2]^{\frac{1}{2}n-q}} = \frac{2\theta_1\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n-q)\,\Gamma(q)} \int_\varepsilon^\infty \frac{s^{q-1}(\theta_1^2-\sigma)^{q-1}\,ds}{(1+s)^{\frac{1}{2}n}},$$

where $d\Sigma$ is the element of the surface $(\xi-\alpha_1)^2+\ldots=\theta_1^2$, and the integration is extended over the entire surface.

A slight change of form is convenient. We have

$$\theta_1^2-\sigma=\theta_1^2-\frac{k^2}{1+s}-\frac{v^2}{s}=\frac{1}{s(1+s)}(\theta_1^2s^2+\chi s-v^2),$$

if we suppose $\chi=\theta_1^2-k^2-v^2$.

The formulæ become

$$\int \frac{d\xi\ldots}{[(\xi-\alpha)^2\ldots+v^2]^{\frac{1}{2}n-q}} = \frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n-q)\,\Gamma(q+1)} \int_\varepsilon^\infty \frac{(\theta_1^2s^2+\chi s-v^2)^q\,ds}{s\,(1+s)^{\frac{1}{2}n+q}},$$

$$\int \frac{d\Sigma}{[(\xi-\alpha)^2\ldots+v^2]^{\frac{1}{2}n-q}} = \frac{2\pi^{\frac{1}{2}n}\theta_1}{\Gamma(\frac{1}{2}n-q)\,\Gamma q} \int_\varepsilon^\infty \frac{(\theta_1^2s^2+\chi s-v^2)^{q-1}\,ds}{(1+s)^{\frac{1}{2}n+q-1}},$$

in which ε is the positive root of the equation

$$\theta_1^2\varepsilon^2+\chi\varepsilon-v^2=0.$$

I propose to transform these formulæ by means of the theory of images; it will be convenient to investigate some preliminary formulæ. Suppose $\lambda^2=\alpha^2+\beta^2\ldots$, $\lambda_1^2=\alpha_1^2+\beta_1^2\ldots$. Also consider the new constants $a, b,\ldots, a_1, b_1\ldots u, f_1$, determined by the equations

$$\frac{\delta^2\alpha}{\lambda^2+v^2}=a, \qquad \frac{\delta^2\alpha_1}{\lambda_1^2-\theta_1^2}=a_1,$$
$$\vdots \qquad\qquad \vdots$$
$$\frac{\delta^2 v}{\lambda^2+v^2}=u, \qquad \frac{\delta^2\theta_1}{\lambda_1^2-\theta_1^2}=f_1,$$

where δ is arbitrary. Then, putting

$$l^2=a^2+b^2\ldots, \quad l_1^2=a_1^2+b_1^2\ldots,$$

it is easy to see that

$$(\lambda^2+v^2)(l^2+u^2)=\delta^4, \quad (\lambda_1^2-\theta_1^2)(l_1^2-f_1^2)=\delta^4$$

and

$$\frac{\delta^2 a}{l^2+u^2}=\alpha, \qquad \frac{\delta^2 a_1}{l_1^2-f_1^2}=\alpha_1,$$
$$\vdots \qquad\qquad \vdots$$
$$\frac{\delta^2 u}{l^2+u^2}=v, \qquad \frac{\delta^2 f_1}{l_1^2-f_1^2}=\theta_1.$$

Proceeding to express the single integrals in terms of the new constants, we have in the first place $k^2 = \delta^4 k^2$, where

$$k^2 = \left(\frac{a}{l^2+u^2} - \frac{a_1}{l_1^2 - f_1}\right)^2 + \ldots;$$

or if we write

$$aa_1 + bb_1 \ldots = ll_1 \cos\omega,$$

we have

$$k^2 = \frac{l^2}{(l^2+u^2)^2} + \frac{l_1^2}{(l_1^2 - f_1^2)^2} - \frac{2ll_1\cos\omega}{(l^2+u^2)(l_1^2-f_1^2)}.$$

Hence also $\chi = \delta^4 j$, where

$$j = \frac{f_1^2}{(l_1^2 - f_1^2)^2} - k^2 - \frac{u^2}{(l^2+u^2)^2},$$

whence

$$-j = \frac{1}{l^2+u^2} + \frac{1}{l_1^2 - f_1^2} + \frac{2ll_1\cos\omega}{(l^2+u^2)(l_1^2-f_1^2)},$$

$$= \frac{1}{(l^2+u^2)(l_1^2-f_1^2)}[p^2 + u^2 - f_1^2],$$

where $p^2 = l^2 + l_1^2 - 2ll_1\cos\omega$, that is

$$p^2 = (a-a_1)^2 + (b-b_1)^2 + \ldots$$

Consequently $\theta_1^2 s^2 + \chi s - v^2 = \delta^4 \Pi$, where Π is given by

$$\Pi = \frac{f_1^2}{(l_1^2 - f_1^2)^2} s^2 - \frac{(p^2 + u^2 - f_1^2)}{(l^2+u^2)(l_1^2 - f_1^2)} s - \frac{u^2}{(l^2+u^2)^2}.$$

And it is clear that ε will be the positive root of

$$0 = \frac{f_1^2}{(l_1^2 - f_1^2)^2}\varepsilon^2 - \frac{(p^2+u^2-f_1^2)}{(l^2+u^2)(l_1^2-f_1^2)}\varepsilon - \frac{u^2}{(l^2+u^2)^2}.$$

And it may be noticed that, in the particular case of $u = 0$, the roots of this equation are 0, and $\frac{(p^2 - f_1^2)(l_1^2 - f_1^2)}{l^2 f_1^2}$. Consequently if $p^2 - f_1^2$ and $l_1^2 - f_1^2$ are of opposite signs, we have $\varepsilon = 0$; but if $p^2 - f_1^2$ and $l_1^2 - f_1^2$ are of the same sign, $\varepsilon = \frac{(p^2 - f_1^2)(l_1^2 - f_1^2)}{l^2 f_1^2}$.

In order to transform the double integrals, considering the new variables $x, y \ldots$ I write $x^2 + y^2 \ldots = r^2$ and

$$\xi = \frac{\delta^2 x}{r^2}, \ldots$$

whence also, if $\xi^2 + \eta^2 + \ldots = \rho^2$ (which gives $r\rho = \delta^2$), we have

$$x = \frac{\delta^2 \xi}{\rho^2} \ldots$$

Also it is immediately seen that

$$(\xi - \alpha)^2 + \ldots v^2 = \frac{\delta^4}{(l^2 + u^2)\, r^2} \{(x - a)^2 + \ldots u^2\},$$

$$(\xi - \alpha_1)^2 \ldots - \theta_1^2 = \frac{\delta^4}{(l_1^2 - f_1^2)\, r^2} \{(x - a_1)^2 + \ldots - f_1^2\}.$$

And from the latter equation it follows that the limiting condition for the first integral is $(x - a_1)^2 + \ldots \gtreqless f_1^2$ (there is no difficulty in seeing that the sign $<$ in the former limiting condition gives rise here to the sign $>$), and that the second integral has to be extended over the surface $(x - a_1)^2 + \ldots = f_1^2$. Also if dS represent the element of this surface, we may obtain

$$d\xi\, d\eta \ldots = \frac{\delta^{2n}}{r^{2n}}\, dx\, dy \ldots, \quad d\Sigma = \frac{\delta^{2n-2}}{r^{2n-2}}\, dS;$$

and, combining the above formulæ,

$$\int \frac{dx\, dy \ldots}{(x^2 + y^2 \ldots)^{\frac{1}{2}n+q} \{(x - a)^2 + (y - b)^2 \ldots + u^2\}^{\frac{1}{2}n-q}}$$

$$= \frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n - q)\, \Gamma(q + 1)\, (l^2 + u^2)^{\frac{1}{2}n-q}} \int_\varepsilon^\infty \frac{\Pi^q ds}{s\,(1 + s)^{\frac{1}{2}n+q}},$$

the limiting condition of the multiple integral being

$$(x - a_1)^2 + (y - b_1)^2 \ldots \gtreqless f_1^2,$$

and

$$\int \frac{dS}{(x^2 + y^2 \ldots)^{\frac{1}{2}n+q-1} \{(x - a)^2 + (y - b)^2 + u^2\}^{\frac{1}{2}n-q}}$$

$$= \frac{2\pi^{\frac{1}{2}n} f_1}{\Gamma(\frac{1}{2}n - q)\, \Gamma q\, (l^2 + u^2)^{\frac{1}{2}n-q} (l_1^2 \sim f_1^2)} \int_\varepsilon^\infty \frac{\Pi^{q-1} ds}{(1 + s)^{\frac{1}{2}n+q-1}},$$

where dS is the element of the surface $(x - a_1)^2 + (y - b_1)^2 \ldots = f_1^2$ and the integration extends over the entire surface.

And, recapitulating,

$$l^2 = a^2 + b^2 + \ldots, \quad l_1^2 = a_1^2 + b_1^2 + \ldots, \quad p^2 = (a - a_1)^2 + (b - b_1)^2 + \ldots,$$

$$\Pi = \frac{f_1^2}{(l^2 - f_1^2)^2}\, s^2 - \frac{(p^2 + u^2 - f_1^2)}{(l_1^2 - f_1^2)(l^2 + u^2)}\, s - \frac{u^2}{(l^2 + u^2)^2}:$$

and ε is the positive root of the equation $\Pi = 0$. The only obviously integrable case is that for which in the second formula $q = 1$; this gives

$$\int \frac{dS}{(x^2+y^2\ldots)^{\frac{1}{2}n}\{(x-a)^2+(y-b)^2+u^2\}^{\frac{1}{2}n-1}} = \frac{2\pi^{\frac{1}{2}n} f_1}{\Gamma(\frac{1}{2}n)\,(l^2+u^2)^{\frac{1}{2}n-1}(l_1^2-f_1^2)(1+\varepsilon)^{\frac{1}{2}n-1}}.$$

In the case of $u = 0$, we have, as before, when $p^2-f_1^2$ and $l_1^2-f_1^2$ are of opposite signs, $\varepsilon = 0$, and therefore $1+\varepsilon = 1$; but when $p^2-f_1^2$ and $l_1^2-f_1^2$ are of the same sign, the value before found for ε gives

$$1+\varepsilon = \frac{1}{l^2 f_1^2}\{l^2 f_1^2 + (p^2-f_1^2)(l_1^2-f_1^2)\}.$$

Consider the image of the origin with respect to the sphere $(x-a_1)^2+(y-b_1)^2\ldots = f_1^2$, the coordinates of this image are

$$\frac{a_1}{l_1^2}(l_1^2-f_1^2),\quad \frac{b_1}{l_1^2}(l_1^2-f_1),\ldots$$

Consequently, if μ be the distance of this image from the point $(a, b\ldots)$, we have

$$\mu^2 = \left\{a - \frac{a}{l_1^2}(l_1^2-f_1^2)\right\}^2 + \ldots$$

$$= \frac{1}{l_1^2}\{l^2 f_1^2 + (p^2-f_1^2)(l_1^2-f_1^2)\};$$

whence, by a simple reduction,

$$1+\varepsilon = \frac{l_1^2\mu^2}{l^2 f_1^2},$$

or the values of the integral are

$p^2-f_1^2$ and $l_1^2-f_1^2$ opposite signs, $I = \dfrac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}\dfrac{f_1}{l^{n-2}(l_1^2-f_1^2)}$,

$p^2-f_1^2$ and $l_1^2-f_1^2$ the same sign, $I = \dfrac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}\dfrac{f_1^{n-1}}{l_1^{n-2}\mu^{n-2}(l_1^2-f_1^2)}$,

where μ is the distance from the point $(a, b\ldots)$ of the image of the origin with respect to the sphere $(x-a_1)^2+\ldots-f_1^2 = 0$.

No. 2, Stone Buildings,
August 6, 1850.

ON THE PRINCIPLES OF THE CALCULUS OF FORMS.

By J. J. Sylvester, Barrister-at-Law.

PART I. SECT. IV.—*Reciprocity, also Properties and Analogies of certain Invariants, &c.*

It will hereafter be found extremely convenient to represent all systems of variables cogredient with the original system in the primitive form by letters of the Roman, and all contragredient systems by letters of the Greek alphabet; the rules for concomitance may then be applied without paying any regard to the distinction between the direction of the march of the substitutions, the variables at the close of each operation as it were telling their own tale in respect of being cogredients or contragredients. This distinction has not (as it should have been) been uniformly observed in the preceding sections; as, for instance, in the notation for emanants which have been derived by the application of the symbol $\left(\xi \frac{d}{dx} + \eta \frac{d}{dy} + \&c.\right)^2$, instead of the more appropriate one $\left(x' \frac{d}{dx} + y' \frac{d}{dy} + \&c.\right)^2$.

The observations in this section will refer exclusively to points of doctrine which have been started in the preceding sections in such order as they more readily happen to present themselves. And, first, as to some important applications of the reciprocity method referred to in notes (6) and (8) of the Appendix.

The practical application of this method will be found greatly facilitated by the rule that x, y, z, &c. may always in any combination of concomitants be replaced respectively by $\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}$, &c., and *vice versâ*. I shall apply this prolific principle of reciprocity to elucidate some of the properties and relations of Aronhold's S and T, and certain other kindred forms. This S and T are the quartinvariant and sextinvariant respectively of a cubic of three variables. I give the names of s and t to the quadrinvariant and cubinvariant of the quartic function of two variables. Furthermore, whoever will consider attentively the remarks made in Section II. of the foregoing relative to reciprocal polars, will apprehend without any difficulty that to every invariant of a function of any degree of any number of variables will correspond a contravariant of a function of the same degree of variables one more in number, and that between such

contravariants, whatever relations exist expressed independently of all other quantities, precisely the same relations must exist between the corresponding contravariants. Thus, then, to s and t the two invariants of $(x, y)^4$ will correspond two contravariants σ and θ two contravariants to $(x, y, z)^4$ and to S and T the two invariants of $(x, y, z)^3$ will correspond Σ and ϑ two contravariants of $(x, y, z, t)^3$. Calling r the resultant of $(x, y)^4$, R the resultant of $(x, y, z)^3$, ρ the polar reciprocal, or, more briefly, the reciprocant of $(x, y, z)^4$, and (R) the reciprocant of $(x, y, z, t)^3$, we have the following equations (presuming that all the quantities are previously affected with the proper numerical multipliers), viz.

$$r = s^3 + t^2, \qquad \rho = \sigma^3 + \tau^2,$$
$$R = S^3 + T^2, \quad (R) = \Sigma^3 + \vartheta^2,$$

I propose in this First Annotation to point out the remarkable analogies which exist between the modes of generating the four pairs of quantities s, t, &c., the functions severally corresponding to which I shall call u, ω, U, Ω. The Hessian corresponding to any of these functions will be denoted by an H prefixed, and when we have to consider, not the pure Hessian, but the matrix formed from it by adding a vertical and horizontal border of variables, the same in number but contragredient to the variable of the function (as, for instance, the Hessian of u bordered with ξ, η horizontally and vertically, or of U with ξ, η, ζ), then I shall denote the result by the prefix symbol $\overline{H}$, and if there be occasion to add two borders, as ξ, η, ζ; ξ', η', ζ', both repeated in the horizontal and vertical direction, the result will be typified by the prefix $\overline{\overline{H}}$.

Now, in the first place, as observed by me in note (8) of the Appendix in the last number; if we call the coefficients of U (10 in number) a, b, c, d, &c., we have

$$S = \overline{H}\left\{\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}; \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right\} \overline{H}\{x, y, z; \xi, \eta, \zeta\},$$

also $$T = \frac{dS}{da}\cdot\frac{d^3H}{dx^3} + \frac{dS}{db}\cdot\frac{d^3H}{d^2x.dy} + \frac{dS}{dc}\cdot\frac{d^3H}{d^2xdz} + \&c.$$

I will now add the further important relation

$$S^2 = \frac{dT}{da}\cdot\frac{d^3H}{dx^3} + \frac{dT}{db}\cdot\frac{d^3H}{d^2xdy} + \frac{dT}{dc}\cdot\frac{d^3H}{d^2xdz} + \&c.^*$$

* It will be found hereafter convenient to designate contravariants formed in this manner from invariants as *Evects* of such invariants or

so that it will be observed if all the derivatives of S are zero, T is zero, and *vice versâ*.

Precisely in the same way, using h and $\overline{\mathrm{h}}$ to denote respectively the Hessian of u and the same bordered with ξ, η, we have

$$s = \overline{\mathrm{h}}\left(\frac{d}{d\xi}, \frac{d}{d\eta}; \frac{d}{dx}, \frac{d}{dy}\right)\overline{\mathrm{h}}(x, y; \xi, \eta),$$

$$t = \frac{ds}{da}\cdot\frac{d^4h}{dx^4} + \frac{ds}{db}\cdot\frac{d^4h}{dx^3.dy} + \frac{ds}{dc}\cdot\frac{d^4h}{dx^2.dy^2} + \&c.$$

$$s^2 = \frac{dt}{da}\cdot\frac{d^4h}{dx^4} + \frac{dt}{db}\cdot\frac{d^4h}{dx^3.dy} + \frac{dt}{dc}\cdot\frac{d^4h}{dx^2.dy^2} + \&c.$$

Again, taking $(\overline{\overline{H}})$ the second bordered Hessian of Ω; that is, Ω bordered as well horizontally as vertically with the double lines and columns $\xi, \eta, \zeta, \theta; \xi', \eta', \zeta', \theta'$,

$$\Sigma = (\overline{\overline{H}})\left(\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}, \frac{d}{d\theta}; \frac{d}{dx}, \frac{d}{dy}, \frac{d}{d\theta}, \frac{d}{dz}; \xi', \eta', \zeta', \theta'\right)$$
$$.(\overline{\overline{H}})(x, y, z, t; \xi, \eta, \zeta, \theta; \xi', \eta', \zeta', \theta'),$$

$$\vartheta = \frac{d\Sigma}{da}\cdot\frac{d^3\overline{H}}{dx^3} + \frac{d\Sigma}{db}\cdot\frac{d^3\overline{H}}{dx^2.dy} + \frac{d\Sigma}{dc}\cdot\frac{d^3\overline{H}}{dx^2.dz} + \frac{d\Sigma}{dd}\cdot\frac{d^3\overline{H}}{dx^2.dt} + \&c.,$$

$$\Sigma^2 = \frac{d\vartheta}{da}\cdot\frac{d^3\overline{H}}{dx^3} + \frac{d\vartheta}{db}\cdot\frac{d^3\overline{H}}{dx^2.dy} + \&c.$$

In like manner again

$$\sigma = (\overline{\overline{h}})\left\{\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}; \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}; \xi', \eta', \zeta'\right\}$$
$$.\overline{\overline{h}}\{x, y, z; \xi, \eta, \zeta; \xi', \eta', \zeta'\},$$

$$\theta = \frac{d\sigma}{da}\cdot\frac{d^4.(\overline{h})}{dx^4} + \&c.,$$

$$\sigma^2 = \frac{d\theta}{da}\cdot\frac{d^4.\overline{h}}{dx^4} + \&c.,$$

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contravariants, and according to the number of times that such process of derivation is applied, 1st, 2nd, 3rd, &c. evects. Such evects form a peculiar class, and when considered generally, without reference to the base to which they refer, they may be termed evectants. Evectants will be again distinguishable according as their base is an invariant simply or a contravariant. Perhaps the terms pure and affected evectants may serve to mark this distinction. For some further remarks on evectants see Note (1).

σ and ϑ are the same quantities as are calculated by Mr. Salmon, in his inestimable work *On Higher Plane Curves*, but are there expressed under the names of S and T, with the sole difference that in place of x, y, z, used by Mr. Salmon, the contragredient variables ξ', η, ζ' are used in the expressions above. Mr. Salmon has also pointed out to me that σ may be obtained by operating with

$$\left(\xi^4 \frac{d}{da} + \xi^3\eta \frac{d}{db} + \xi^3\zeta \frac{d}{dc} + \&c.\right)$$

directly upon I a cubic invariant of the function u, or $(x, y, z)^4$. This I is no other than the simple commutant obtained by operating upon u with the commutantive symbol formed by taking four times over the line $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, agreeable to the remark made in the third section that every function of an even degree of n variables possesses an invariant of the n^{th} order in extension of Mr. Cayley's observation that every such function of two variables possesses a quadrinvariant, *i.e.* an invariant of the second order.

I need hardly remark that σ is of 2 dimensions in the coefficients and of 4 in the contragredient variables, τ of 3 in the coefficients and of 5 in the contragredients, Σ of 4 in the constants and 4 in the contragredients, ϑ of 6 in the constants and 6 in the contragredients, or that the single-bordered Hessians of u and U and the double-bordered Hessians of ω and Ω are each of them quadratic in respect of the x &c. as well as of the ξ &c. systems.

If the right numerical factors be attributed to S, T, Aronhold has shewn that

$$H.H(U) + T.H(U) + S^2.U = 0,$$

and in my paper in the last May No., I gave the equation

$$h.h(u) + s.h(u) + t(u) = 0.$$

I think it highly probable that it will be found that the analogous equations obtain, viz.

$$(\overline{H})(\overline{H})\,\Omega + \vartheta.\overline{H}\Omega + \Sigma^2\Omega = 0,$$

$$(\bar{h})(\bar{h})\,\omega + \sigma.(\bar{h}).\omega = \vartheta\omega = 0.$$

These remarkable equations, if verified, (of which I can scarcely doubt) will be most powerful aids to the dissection of the forms ω, Ω, and thereby to the detection of the

fundamental properties of curves of the fourth and surfaces of the third degree, of which at present so little is known. It will have been observed that in the preceding developments the contravariants of ω and Ω were derived in precisely the same way from ω and Ω as the corresponding invariants of u and U from u and U, with the sole difference that the Hessian used in the two latter cases is replaced by a single-bordered Hessian in the two former cases, and a single-bordered Hessian in the two latter by a double-bordered Hessian in the two former. The analogies are not even yet stated exhaustively; for it will be remembered (as shewn in the third section), that T and S can be derived directly and concurrently by means of operating with the commutantive symbol

$$\left|\begin{matrix} \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz} \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \end{matrix}\right| \text{ upon } \overline{\overline{H}}(U) + \lambda(x\xi + y\eta + z\zeta)^2,$$

which gives a result of the form $m\{\lambda^3 + S\lambda + T\}$, m being a number, and I conjecture that if

$$\begin{matrix} \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, & \frac{d}{dt} \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, & \frac{d}{dt} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta}, & \frac{d}{d\theta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta}, & \frac{d}{d\theta} \end{matrix}$$

be made to operate upon

$$\overline{\overline{H}}\Omega + \lambda(x\xi + y\eta + z\zeta + t\theta)^2,$$

and the result be put under the form

$$m(\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D),$$

that A will be zero, B and C will be respectively Σ and ϑ, and perhaps D (a contravariant, if it effectively exist, of

8 dimensions in the coefficients of Ω, and of a like number in the contragredients ξ', η', ζ', θ'), also zero. But of the evanescence of D I do not speak with any degree of assurance.

Mr. Salmon has made an excellent observation to the effect that if we call (σ) what σ becomes when ξ', η', ζ' are replaced by $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, $(\sigma)\,h(\omega)$ will represent a covariant to ω of $3+2$, that is, 5 dimensions in the coefficients, and of $6-4$, that is, of 2 dimensions in x, y, z, $h(\omega)$ being of 3 and 6 dimensions in these respectively, and σ of 2 and 4 dimensions respectively in the same. Now these resulting dimensions 5 and 2 precisely agree with the form especially noticed by me in Note (2) of the Appendix, where it was derived as one of a group by the method of unravelment.* There can be little doubt that these two conics each of them indissolubly connected with every curve of the fourth degree are identical. The form $(\sigma)\,h(\omega)$ enables us to prove readily (thanks to Mr. Salmon's calculation of σ, given in his *Higher Plane Curves*, p. 101, under the name of S) that this is a *bonâ fide* existent conic.

For if we take a particular case of ω, say

$$\omega = a_1x^4 + b_2y^4 + c_3z^4 + 6dy^2z^2,$$

we find

$$h(\omega) = \begin{vmatrix} a_1x^2 & 0 & 0 \\ 0 & b_2y^2 + dz^2 & dyz \\ 0 & dyz & c_3z^2 + dy^2 \end{vmatrix}$$

$$= a_1(b_2c_3 + d^2)\,x^2y^2z^2 + a_1b_2.d.x^2y^4 + a_1c_3dx^2z^4,$$

and σ becomes $\qquad a_1d\eta^2\zeta'^2,$

and consequently (σ) is

$$a_1d\left(\frac{d}{dy}\right)^2.\left(\frac{d}{dz}\right)^2,$$

and therefore $\quad (\sigma)\,h(\omega) = 4a_1^2d(b^2c^2 + d^2)\,x^2,$

the conic here reducing to a pair of coincident straight lines. This example demonstrates that the conic is in general actually existent.

As I have said so much upon S and T it may not be irrelevant to state in this place how I obtained the conditions for U, the characteristic of the curve of the third

* Vide Note (2).

degree becoming the characteristic of a conic and a straight line, *i.e.* breaking up into a linear and a quadratic factor, which Mr. Salmon has inserted in the notes to his work above referred to. When U is of this form it may obviously by linear transformations be expressed by $ax^3 + bdxyz$, but when starting with the general form,

$$a_1x^3 + b_2y^3 + c_3z^3 + \&c. + 6Dxyz,$$

we form two contravariants from S and T, to wit

$$\left(\xi^3 \frac{d}{da_1} + \eta^3 \frac{d}{db_2} + \zeta^3 \frac{d}{dc_3} + \&c. + \xi\eta\zeta \frac{d}{dD}\right) S, \text{ say } S',$$

$$\left(\xi^3 \frac{d}{da_1} + \eta^3 \frac{d}{db_2} + \zeta^3 \frac{d}{dc_3} + \&c. + \xi\eta\zeta \frac{d}{dD}\right) T, \text{ say } T',$$

and then make $a_1 = a$, $D = d$, and all the other coefficients zero, it will easily be seen on examining the forms of S and T, given by Mr. Salmon (pp. 184, 186), that (S) and (T) (the evectants of S and T) become respectively

$$4d^3\xi\eta\zeta, \quad 31d^5\xi\eta\zeta;$$

we have therefore $(T) + \lambda(S) = 0$: and (T) and (S), although contravariantive to their primitive U, are covariantive with one another, so that $(T) + \lambda(S) = 0$ is a persistent relation unaffected by linear transformations; it follows therefore that when U is of, or reducible to, the form supposed,

$$\frac{dS}{da_1} : \frac{dS}{db_2} : \frac{dS}{dc_3} : \&c. : \frac{dS}{dD}$$
$$= \frac{dT}{da_1} : \frac{dT}{db_2} : \frac{dT}{dc_3} : \&c. : \frac{dT}{dD},$$

which is the criterion given in the note referred to.*

I am also able to obtain these equations more directly by another method found upon a New View of the Theory of Elimination, an account of which, however, I must reserve for another occasion, but which, I may mention, serves to fix not merely the conditions, as in the ordinary restricted theory, that a given set of equations may be simultaneously satisfiable by some one system of values of the variables, but the *conditions* that such set of equations may be simultaneously satisfiable by any given number of distinct systems of variables.

* Mr. Salmon has remarked that the two evectants (S) and (T) intersect in the nine cuspidal points of the polar reciprocal to the curve.

Mr. Salmon has remarked to me to the effect that if in τ we write $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, in place of the contragredients, and call τ so altered (τ), then $(\tau)h(\omega)$ will be an invariant of 6 dimensions in the coefficients of ω. This sext-invariant I have little doubt is identical with that obtained by operating upon ω with the commutation symbol

$$\left.\begin{array}{cccccc} \left(\frac{d}{dx}\right)^2, & \frac{d}{dx}\cdot\frac{d}{dy}, & \left(\frac{d}{dy}\right)^2, & \frac{d}{dy}\cdot\frac{d}{dz}, & \left(\frac{d}{dz}\right)^2, & \frac{d}{dz}\cdot\frac{d}{dx} \\ \left(\frac{d}{dx}\right)^2, & \frac{d}{dx}\cdot\frac{d}{dy}, & \left(\frac{d}{dy}\right)^2, & \frac{d}{dy}\cdot\frac{d}{dz}, & \left(\frac{d}{dz}\right)^2, & \frac{d}{dz}\cdot\frac{d}{dx} \end{array}\right\}$$

This, like every other commutant of 2 lines only, is of course capable of being expressed under the form of an ordinary determinant, and the remark is not without interest, as shewing how the proposition known with respect to quadratic functions of any number of variables, viz. of every such having an invariantive determinant, lends itself to the general case of functions of any even degree of any number of variables which also have always an invariantive determinant attached to them, of which the terms are simple coefficients of such functions. The only peculiarity (if it be one) of quadratic functions in this respect being that they have each but one invariant of such form and no other. In the case before us, if we write

$$\begin{aligned}\omega = {} & a_1x^4 + b_2y^4 + c_3z^4 + 4a_2x^3y + 4a_3x^3z + 4b_1y^3x + 4b_3y^3z \\ & + 4c_1z^3x + 4c_2z^3y + 6dy^2z^2 + 6ez^2x^2 + 6fx^2y^2 + 12lx^2yz \\ & + 12mxy^2z + 12nxyz^2,\end{aligned}$$

the sext-invariant in question becomes representable under the form of the determinant

$$\begin{array}{cccccc} a_1 & a_2 & f & l & e & a_3 \\ a_2 & f & b_1 & m & n & l \\ f & b_1 & b_2 & b_3 & d & m \\ l & m & b_3 & d & c_2 & n \\ e & n & d & c_2 & c_3 & c_1 \\ a_3 & l & m & n & c_1 & e \end{array}^*$$

* This determinant is identical with the determinant formed by taking the second differential coefficients of the function and arranging in the usual manner the coefficients of the several powers and combinations of powers of the variables treated as if they were independent quantities.

Before quitting the subject of S and T the two invariants of the cubic function of 3 variables, or, as it may be termed, of the cubic curve, it may not be amiss to give the complete table which I have formed corresponding to all the singular cases which can befall such curve, which will be seen below to be eight in number; it is of the highest importance to push forward the advanced posts of geometry, and for this purpose to obtain the same kind of absolute power and authority over, and clear and absolute knowledge of, the properties and affections of cubic forms as have been already attained for forms of the second degree.

Let $U = ax^3 + 4bx^2y + 4cx^2z + \&c.$

(1). When U has one double point $S^3 + T^2 = 0$.

(2). When U has two double points, *i.e.* becomes a conic and right line

$$\frac{dS}{da} \cdot \frac{dT}{db} - \frac{dS}{db} \cdot \frac{dT}{da} = 0, \ \&c. \ \&c.$$

(3). When U has a cusp $S = 0, \ T = 0$.

(4). When U has two coincident double points, *i.e.* is a conic and a tangent line thereto, which comprises the two preceding cases in one,

$$\frac{dT}{da} = 0, \quad \frac{dT}{db} = 0, \ \&c.$$

and also *therefore* $S = 0$.

(5). When U becomes three right lines forming a triangle

$$\frac{d^2S}{da.db} \cdot \frac{d^2T}{dc.de} - \frac{d^2T}{da.db} \cdot \frac{d^2S}{dc.de} = 0, \ \&c.$$

where a, b, c, e each represent any of the coefficients arbitrarily chosen, whether distinct or identical.

Another, and lower in degree system of equations, may be substituted for the above, obtained by affirming the equality of the ratios between the coefficients of U and the corresponding coefficients of its Hessian.

(6). When U represents a pencil of three rays meeting in a point

$$\frac{dS}{da} = 0, \quad \frac{dS}{db} = 0, \ \&c.$$

and also *therefore* $T = 0$.

Also in place of this system may be substituted the system obtained by taking all the coefficients of the Hessian zero.

(7). When (U) becomes a line, and two other coincident lines,

$$\frac{dS}{da} = 0, \quad \frac{dS}{db} = 0, \text{ \&c.}$$

and also $$\frac{d^2T}{da} = 0, \quad \frac{dT}{da.db} = 0, \text{ \&c.}$$

I have not ascertained whether this second system necessarily implies the first; I rather think that it does not. In the preceding case also it would be interesting to shew the direct algebraical connexion between the system formed by the coefficients of the Hessian and the system consisting of the first derivatives of S.

(8). When U becomes a perfect cube representing three coincident right lines

$$\frac{d^2S}{da^2} = 0, \quad \frac{d^2S}{da.db} = 0, \text{ \&c.}$$

and $$\frac{d^2T}{da^2} = 0, \quad \frac{d^2T}{da.db} = 0, \text{ \&c.}$$

The first of these systems of equations necessarily implies the equations $\frac{dT}{da} = 0$, $\frac{dT}{db} = 0$, &c., as is obvious from the equation

$$T = \frac{dS}{da} \cdot \frac{d^3H}{dx^3} + \frac{dS}{db} \cdot \frac{d^3H}{dx^2.dy}, \text{ \&c.}$$

but not necessarily the second and lower system $\frac{d^2T}{da^2} = 0$, &c. above written.

So if we take

$$u = ax^4 + 4bx^3y + 6ca^2y^2 + 4dxy^3 + ey^4:$$

when 2 roots are equal $s^3 + t^2 = 0$,

when 2 pairs of roots are equal

$$\frac{ds}{da} \cdot \frac{dt}{db} - \frac{ds}{db} \cdot \frac{dt}{da} = 0, \text{ \&c.}$$

when 3 roots are equal $s = 0$, $t = 0$,

and when all 4 roots are equal

$$\frac{dt}{da} = 0, \quad \frac{dt}{db} = 0, \text{ \&c.}$$

Before closing this Section I may make a remark, in reference to the sextic invariant of ω, which admits of being extended to all commutants formed by operating upon the function with a commutantive symbol obtained by writing over one another lines consisting of powers of $\frac{d}{dx}$, $\frac{d}{dy}$, &c. and their combinations (to which, in the third section, I gave the name of *compound* commutants, a qualification which, for reasons that will hereafter be adduced, I think it advisable to withdraw). The remark I have to make is this, viz. that the invariant obtained by operating upon ω with

$$\left.\begin{matrix} \left(\frac{d}{dx}\right)^2 \frac{d}{dx}\cdot\frac{d}{dy} \left(\frac{d}{dy}\right)^2 \frac{d}{dy}\cdot\frac{d}{dz} \left(\frac{d}{dz}\right)^2 \frac{d}{dz}\cdot\frac{d}{dx} \\ \left(\frac{d}{dx}\right)^2 \frac{d}{dx}\cdot\frac{d}{dy} \left(\frac{d}{dy}\right)^2 \frac{d}{dy}\cdot\frac{d}{dz} \left(\frac{d}{dz}\right)^2 \frac{d}{dz}\cdot\frac{d}{dx} \end{matrix}\right\}$$

is precisely the same as may be obtained by operating with

$$\left.\begin{matrix} \frac{d}{du}, \frac{d}{dv}, \frac{d}{dw}, \frac{d}{dp}, \frac{d}{dq}, \frac{d}{dr} \\ \frac{d}{du}, \frac{d}{dv}, \frac{d}{dw}, \frac{d}{dp}, \frac{d}{dq}, \frac{d}{dr} \end{matrix}\right\}$$

upon the concomitant quadratic function to ω obtained by the method of unravelment, as in Note (2) of the Appendix, (p. 93); and so, in general, every commutant obtained by operating upon a function of any number of variables of the degree $2mp$ with a symbol consisting of $2p$ lines in which the m^{th} powers of $\frac{d}{dx}, \frac{d}{dy}$, &c. and their m^{th} combinations occur, will be identical with the commutant obtained by operating with a symbol also of $2p$ lines, in which only the simple powers occur of $\frac{d}{du}$, $\frac{d}{dv}$, &c. (where u, v, &c. are cogredient with x^p, $x^{p-1}y$, &c.) upon a function of u, v, &c., formed by the method of unravelment from the given function.

Finally, before quitting the subject of reciprocity, I may state, it follows from the general statement made at the commencement of this Section, that inasmuch as

$$(x\xi + y\eta + z\zeta, \text{ &c.})^2$$

is a universal concomitant form, so also must

$$\left(\frac{d}{d\xi}\cdot\frac{d}{dx} + \frac{d}{d\eta}\cdot\frac{d}{dy} + \frac{d}{d\zeta}\cdot\frac{d}{dz}, \text{ &c.}\right)^2$$

be an universal concomitant symbol of operation; accordingly it is certain that any concomitant in which x, y, z, &c., ξ, η, ζ, &c. enter, operated upon with this symbol, will remain a concomitant: in several cases which I have examined, the effect of this operation will be to produce an evanescent form, but I see no ground for supposing that this is other than an accidental, or at all events for supposing that it is a necessary and universal consequence of the operation. It may also be observed that in the case of as many cogredient sets of variables as variables in each set, as for instance 3 sets of 3 variables each, the determinant which may be formed by arranging them in regular order, as $\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix}$ is evidently a universal concomitant, and moreover an equivocal concomitant, possessing the property of remaining a concomitant when the variables are respectively but simultaneously exchanged for their contragredients ξ, η, ζ; ξ', η', ζ'; ξ'', η'', ζ''; which shews also that in place of the variables may be written the differential operators

$$\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}; \frac{d}{dx'}, \frac{d}{dy'}, \frac{d}{dz'}; \frac{d}{dx''}, \frac{d}{dy''}, \frac{d}{dz''}:$$

a remark which leads us to see the exact place in the general theory occupied by Mr. Cayley's method of generating covariants given in the concluding paragraph of the first section, page 59. I may likewise add, that inasmuch as $(x'\xi + y'\eta + z'\zeta, \text{\&c.})^2$ is a universal concomitant,

$$\left(x' \frac{d}{dx} + y' \frac{d}{dy} + \text{\&c.}\right)^r$$

will be so too, by virtue of the general law of interchange, which conducts immediately to the theory of emanation, shewing that this last symbol, operating upon any function, furnishes covariants thereunto for any integer value of z.

One additional interesting remark presents itself to be made concerning U, the cubic function of x, y, z, which is, that calling as before T its sextic invariant, and $a, 3b, 3c, d$, &c. the coefficient,

$$\left(\xi^3 \frac{d}{da} + \xi^2\eta \frac{d}{db} + \xi^2\zeta \frac{d}{dc} + \xi\eta\zeta \frac{d}{d.d} \text{\&c.}\right)^2 . T$$

will give the polar reciprocal, or, as it has been agreed to term it, the reciprocant of U. I believe the remark of the probability of this being the case originated with myself, but Mr. Cayley, first verified it by actual calculation, using for that

purpose the value of T, given by Mr. Salmon in his work *On the Higher Plane Curves*, already frequently alluded to, and which is an indispensable manual equally for the objects of the higher special geometry as for the new or universal algebra, being in fact a common ground where the two sciences meet and render mutual aid.

Mr. Salmon also observed, that the first evect of T, viz. $\left(\xi^3 \frac{d}{da} + \xi^2\eta \frac{d}{db} \text{ \&c.}\right) T$, was identical in form with what may be termed the first devect of the polar reciprocal, *i.e.* the result of operating upon the polar reciprocal with what U becomes when $\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}$, are substituted in the stead of x, y, z. And inasmuch as, by Euler's law,

$$\left\{a\left(\frac{d}{d\xi}\right)^3 + 3b\left(\frac{d}{d\xi}\right)^2 \cdot \frac{d}{d\eta} \text{ \&c.}\right\} \times \left\{\xi^3 \frac{d}{da} + \xi^2\eta \frac{d}{db} \text{ \&c.}\right\} T$$
$$= 6\left\{a \frac{d}{da} + b \frac{d}{db} + \text{\&c.}\right\} T = 36\,T,$$

it follows that T is the second *devect* of the polar reciprocal, or at least identical with it in point of form. But, since the preceding matter was printed, I have discovered in the course of a most instructive and suggestive correspondence with Mr. Salmon, the principle upon which these and similar identifications depend, thereby dispensing with the necessity for the excessively tedious labour of verification which, even in the simple example before us, would be found to extend over several pages of work.

The theory in which this principle is involved will be given, along with other very important matter, in the next number of the *Journal*.

Supplementary Observations on the Method of Reciprocity.

It has been observed, that ξ, η, &c. may always be inserted in place of $\frac{d}{dx}$, $\frac{d}{dy}$, &c., and *vice versâ*, in a concomitant form, without destroying its concomitance. Accordingly, instead of the evector symbol

$$\xi^3 \frac{d}{da} + \xi^2\eta \frac{d}{db} + \text{\&c.},$$

we may employ

$$\left(\frac{d}{dx}\right)^3 \cdot \frac{d}{da} + \left(\frac{d}{dx}\right)^2 \cdot \frac{d}{dy} \cdot \frac{d}{db} + \text{\&c.};$$

and operating with this upon any concomitant, the result will be a concomitant. Hence we see, for example, that if we take the concomitant $S.H$ formed by the product of the invariant S and the covariant H,

$$\left\{\left(\frac{d}{dx}\right)^3 \frac{d}{da} + \left(\frac{d}{dx}\right)^2 \frac{d}{dy}\,\frac{d}{db} + \&c.\right\} S.H$$

will be a coinvariant; in fact this will be found to be T, the difference between this and the expression before given for T, viz.

$$\left(\frac{d}{dx}\right)^3 H.\,\frac{dS}{da} + \left(\frac{d}{dx}\right)^2 \frac{d}{dy}\,H.\,\frac{dS}{db} + \&c.,$$

being $$S \times \left\{\frac{d}{da}.\left(\frac{d}{dx}\right)^3 H + \frac{d}{db}.\left(\frac{d}{dx}\right)^2.\frac{d}{dy}\,H + \&c.\right\},$$

which is zero, there being no invariant to $(x, y, z)^3$ of the 3rd degree in a, b, c, &c., as the factor multiplied by T would be were it not evanescent. The same observation may be extended to the analogous equations given in pages 180, 181.

I have chiefly, however, made the above observation with a view to making more clear the enunciation of the theorem which I am now about to state, the most important perhaps in its application of any yet brought to light on the subject, but the consequences of which, as I have but quite recently discovered it, must be reserved for a future number of the *Journal*.

Let any function of any number of variables be supposed to have for its coefficients the letters a, b, &c. affected with the ordinary binomial or multinomial coefficients; and let another function be taken identical with the former in all respects, except in the circumstance that all their numerical multipliers are suppressed. Let this function or form be termed the respondent to the primitive: furthermore, by the inverse of any form understand what that form becomes when, in place of x, y, z, &c., ξ, η, ζ, &c.,

$$\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \&c., \quad \frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}, \&c.,$$

are respectively substituted (and so for all the systems of the variables), and likewise at the same time similar substitutions are made of $\frac{d}{da}$, $\frac{d}{db}$, $\frac{d}{dc}$, &c., in place of a, b, c, &c.; then we have this grand and simple law—*The inverse of any con-*

comitant to a respondent is a concomitant to its primitive. When the inverse of any concomitant to the respondent is made to operate upon the same concomitant of the primitive, it will be found that the result is a power of the universal concomitant. If the concomitant to the respondent be an invariant thereof, the rule indicates that on merely replacing in the respondent $a, b, c,$ &c. by $\frac{d}{da}, \frac{d}{db}, \frac{d}{dc}$, &c., the result operating on any invariant or other concomitant of the primitive, leaves it still an invariant or other concomitant. For instance, if we take the function $ax^5 + 5bx^4y + wcx^3y^2 + 10dx^2y^3 + 5ex^2y^4 + fy^5$, which has three invariants L, M, N, of the degrees 4, 8, 12, respectively: and if we call λ, μ, ν what L, M, N become when, in place of a, b, c, d, e, f respectively, we write

$$\frac{d}{da}, \frac{1}{5}\cdot\frac{d}{db}, \frac{1}{w}\cdot\frac{d}{dc}, \frac{1}{10}\cdot\frac{d}{d.d}, \frac{1}{5}\cdot\frac{d}{d.e}\,\frac{d}{d.f},$$

we shall find that $\lambda.M = L$, $\mu N = L$,

and $\lambda.N =$ a linear function of M and L^2.

Again, if in the case of any function of $x, y, z,$ &c., we take, instead of any other concomitant to the respondent, the respondent itself, its inverse gives the symbol of operation

$$\left(\frac{d}{da}\right).\left(\frac{d}{dx}\right)^3 + \frac{d}{db}\left(\frac{d}{dx}\right)^2.\left(\frac{d}{dy}\right), \text{ &c.},$$

just previously treated of. If again, in the case of a function of $x, y,$ say

$$ax^n + nbx^{n-1}.y + \text{&c.} + nb'.xy^{n-1} + a'y^n,$$

we take the inverse of the polar reciprocal of the respondent, we get the operator

$$\frac{d}{da}.\left(\frac{d}{d\eta}\right)^n - b\left(\frac{d}{d\eta}\right)^{n-1}.\frac{d}{d\xi} + \text{&c.};$$

and replacing $\frac{d}{d\eta}, \frac{d}{d\xi}$ by y, x, we find that

$$y^n.\frac{d}{da} - y^{n-1}.x\frac{d}{db}, \text{ &c.},$$

operating on any concomitant, leave it still a concomitant, which is M. Eisenstein's theorem before adverted to, only generalized by the introduction of any concomitant in lieu of the discriminant.

This extraordinary theorem of respondence will be found on reflection to favour the notion of treating the coefficients of a general function as themselves, a system of variables, in a manner contragradient to the terms to which they are affixed.

Finally, there is yet another mode of applying the principle of reciprocity, which must be carefully distinguished from any previously stated in these pages.

I have said that in place of the quantitative symbols of one alphabet, as x, y, z, &c., we may always substitute the operation symbols $\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}$, &c. of the opposite alphabet. But now I say, in place of the quantitative symbols x, y, z, &c. occurring in the concomitant to any form f, may be substituted the quantities (observe, no longer operative symbols but quantities) $\frac{dF}{d\xi}, \frac{dF}{d\eta}, \frac{dF}{d\zeta}$, &c., F being itself any concomitant to f. Thus, for instance, taking F identical with f, we see that $f\left(\frac{df}{d\xi}, \frac{df}{d\eta}, \frac{df}{d\zeta}, \text{&c.}\right)$ is concomitant to f: or again, if f be a function of x, y only, say $f(x, y)$, taking F the polar reciprocal of f, i.e. $f(-\eta, \xi)$, we see that $f\left(-\frac{df}{dx}, \frac{df}{dy}\right)$ will be a concomitant to f: this concomitant, by the way it may be observed, will always contain f as a factor, because when $f = 0$, $x\frac{df}{dx} + y\frac{df}{dy} = 0$. Possibly it may be true that, when f is a function of any number of variables x, y, z, &c., and $F(\xi, \eta, \zeta, \text{&c.})$ its polar reciprocal,

$$f\left(\frac{dF(x, y, z, \text{&c.})}{dx}, \frac{dF(x, y, z, \text{&c.})}{dy}, \text{&c.}\right),$$

which is a concomitant to f, contains f as a factor; but I have not had time to see how this is. It is rather singular that Mr. Cayley and Professor Borchardt of Berlin have both independently made to me the observation that, when $f(x, y)$ is taken a cubic function of x and y, $f\left(\frac{df}{dy}, \frac{-df}{dx}\right)$ is equal to the product of f by the first evectant of the discriminant of f. The general consideration of the consequences of this new and important application of the idea of reciprocity, must be reserved for a future section.

N.B. The reader is requested to take notice that Note (1) referred to in the marginal note at p. 181 is suppressed, and that Note (2) referred to in the marginal note to p. 184 will be given as Note (9), as it has been thought better to make the numbering of the Appendix notes run on from the former sections.

Section V.

Applications and Extension of the Theory of the Plexus.

If $$\phi = ax^4 + 4bx^3y + 6cx^2y^2 - 4dxy^3 + ey^4,$$

we can obtain, by operating catalectically with x', y' upon

$$\left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^2\phi, \quad \left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^4\phi,$$

the two concomitants

$$\begin{vmatrix} ax^2 + 2bxy + cy^2, & bx^2 + 2cxy + dy^2 \\ bx^2 + 2cxy + dy^2, & cx^2 + 2dxy + ey^2 \end{vmatrix} \ldots\ldots (1),$$

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix} \ldots\ldots\ldots\ldots\ldots (2),$$

the one in fact being the Hessian, the other the catalecticant of ϕ itself. Again, if

$$\phi = ax^5 + 5bx^4y + 10cx^3y^2 + \&c. + fy^5,$$

by operating catalectically with x', y' upon the second and fourth emanants, as in the last case, we obtain the two covariants

$$\begin{vmatrix} ax^3 + 3bx^2y + 3cxy^2 + dy^3, & bx^3 + 3cx^2y + 3dxy^2 + ey^3 \\ bx^3 + 3cx^2y + 3dxy^2 + ey^3, & cx^3 + 3dx^2y + 3exy^2 + fy^3 \end{vmatrix} \ldots (1),$$

$$\begin{vmatrix} ax + by, & bx + cy, & cx + dy \\ bx + cy, & cx + dy, & dx + ey \\ cx + dy, & dx + ey, & ex + fy \end{vmatrix} \ldots\ldots (2),$$

which are in fact the Hessian and Canonizant respectively of ϕ. So in general, for a function of x, y of the degree 2ι or $2\iota+1$, we can obtain ι covariantive forms, the first being the Hessian, and the last the catalecticant on the first supposition and the canonizant on the second: calling the index of the function for either case n, the forms appearing in this scale will be of the degree $(r+1)$ in the constants, and of the degree $(r+1)(n-2r)$ in x and y.

In the foot note of page 96 of the present volume of the *Journal*, I intimated that all these determinants admitted of a remarkable transformation.

This transformation may be expressed more elegantly by dealing not directly with the covariant forms as above given, but with their polar reciprocants obtained immediately by writing ξ for $-y$ and η for x.

(1). Suppose $\phi = ax^3 + 2x^2y + 3cxy^2 + dy^3$,

$$\begin{matrix} a & 2b & c \\ b & 2c & d \\ \xi^2 & 2\xi\eta & \eta^2 \end{matrix}$$

will be found to be the reciprocant of its Hessian.

(2). Let $\phi = ax^4 + 4bx^3y + \&c. + ey^4$,

the reciprocant of its Hessian will be found to be

$$\begin{matrix} a & 3b & 3c & d \\ b & 3c & 3d & e \\ \xi^2 & 2\xi\eta & \eta^2 & \\ & \xi^2 & 2\xi\eta & \eta^2 \end{matrix}$$

(3). Let $\phi = ax^5 + 5bx^4y + \&c. + fy^5$,

the reciprocant of its Hessian will be

$$\begin{matrix} a & 4b & 6c & 4d & e \\ b & 4c & 6d & 4e & f \\ \xi^2 & 2\xi\eta & \eta^2 & & \\ & \xi^2 & 2\xi\eta & \eta^2 & \\ & & \xi^2 & 2\xi\eta & \eta^2 \end{matrix}$$

and the reciprocant of its canonizant is

$$\begin{matrix} a & 3b & 3c & d \\ b & 3c & 3d & e \\ c & 3d & 3e & f \\ \xi^3 & 3\xi^2\eta & 3\xi\eta^2 & \eta^3 \end{matrix}$$

The numerical coefficients in this and in the first case are inserted for the sake of uniformity, but it will of course be readily observed that when there is but one line of ξ and η, that the numerical coefficients being the same for each column may be rejected without affecting the form of the result.

So again, if $\phi = ax^6 + 6bx^5y + \&c. + gy^6$,
the reciprocant of the Hessian is

$$\begin{array}{cccccc}
a & 5b & 10c & 10d & 5e & f \\
b & 5c & 10d & 10e & 5f & g \\
\xi^2 & 2\xi\eta & \eta^2 & & & \\
 & \xi^2 & 2\xi\eta & \eta^2 & & \\
 & & \xi^2 & 2\xi\eta & \eta^2 & \\
 & & & \xi^2 & 2\xi\eta & \eta^2
\end{array}$$

and the reciprocant of the second form in the scale, which comes between the Hessian and the catalecticant will be

$$\begin{array}{ccccc}
a & b & c & d & e \\
b & c & d & e & f \\
c & d & e & f & g \\
\xi^3 & \xi^2\eta & \xi\eta^2 & \eta^3 & \\
 & \xi^3 & \xi^2\eta & \xi\eta^2 & \eta^3
\end{array}$$

and so in general. The rule of formation is sufficiently plain not to need formulizing in general terms. It is easy to see that all these forms are concomitants to the function from which they are formed; for example, take

$$\phi = ax^6 + 6bx^5y + \&c. + gy^6,$$

$\left(\frac{d}{dx}\right)^2\phi$, $\frac{d}{dx}\cdot\frac{d}{dy}\phi$, $\left(\frac{d}{dy}\right)^2\phi$ form a plexus.

So likewise if we take $\psi = (x\xi + y\eta)^4$,

$\frac{d\psi}{d\xi}$, $\frac{d\psi}{d\eta}$ form a plexus;

but ψ and ϕ are concomitantive, ψ being a universal concomitant. Hence we may combine together these two plexuses, that is

$$\left.\begin{array}{l}
ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4 \\
bx^4 + 4cx^3y + 6dx^2y^2 + 4exy^3 + fy^4 \\
cx^4 + 4dx^3y + 6ex^2y^2 + 4fxy^3 + gy^4
\end{array}\right\}$$

$$\left.\begin{array}{l}
\xi^3x^4 + 3\xi^2\eta x^3y + 3\xi\eta^2x^2y^2 + \eta^3xy^3 \\
\qquad \xi^3x^3y + 3\xi^2\eta x^2y^2 + 3\xi\eta^2xy^3 + \eta^3y^4
\end{array}\right\}$$

and, by the principle of the plexus, x^4, x^3y, x^2y^2, xy^3, y^4 may be eliminated dialytically, and the resultant will be the determinant last given, which is therefore a contravariant to ϕ.

The manner in which I was led to notice this singular transformation is somewhat remarkable.

In the supplemental part of my essay *On Canonical Forms, &c.*, published by Bell, Fleet Street, my method of solution of the problem of throwing the quintic function of 2 variables under the form $u^5 + v^5 + w^5$, led me to see that u, v, w are the three factors of

$$\begin{matrix} ax + by & bx + cy & cx + dy \\ bx + cy & cx + dy & dx + ey \\ cx + dy & dx + ey & ex + fy \end{matrix}$$

the more simple mode of the solution of the same problem, given by me in the *Philosophical Magazine* for the month of November last, led to

$$\begin{matrix} a & b & c & d \\ b & c & d & e \\ c & d & e & f \\ y^3 & -xy^2 & x^2y & -x^3 \end{matrix}$$

as the product of the same three factors; whence the identity of the two forms become manifest. In the paper last named I gave two proofs, one my own, the other Mr. Cayley's, of a like kind of identity for the canonizant of any odd-degreed function of x, y in general. The proof of the identity of the corresponding forms in the much more general proposition above indicated must be reserved until more pressing and important matters are disposed of. In the foot note referred to, p. 96, line 4, there is an erratum, $2r$ being written in place of $2n$; I ought also to have added, in order to make the sense more clear, that the degree of the catalecticant there referred to in respect of the coefficients would be n.

In note 8 (7th line) ξ, η, ζ in the first G are written in place of $\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}$. I regret to think that there are many other typographical errors in the earlier Sections; the most unfortunate of these is in the note at page 87, in the values of P and Q belonging to the cubic commutant dodecadic function of x and y, the corrected values of which will be given in my next communication. I ought also to observe, in correction of the remark made in the foot note to page 72, that it follows as a consequence of a recent paper by Dr. Hesse in *Crelle's Journal*, that the method given by me in the text applied (according to what I have there termed the 1st process for obtaining an invariant

resembling the resultant) to a system of three cubic equations (in which application only the 1st powers of $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ enter) produces for that case also, as well as for the cases specified in the note, not a counterfeit resemblance of, but the actual resultant itself.

Returning to the theory of the plexus of which I am about to enunciate a most important extension, I beg to refer my readers to the fourth paragraph, p. 60, in the last number of the *Journal*, where I have shewn how to form, under certain conditions, a determinant by combining together various concomitants and eliminating dialytically one set of the variables, which determinant will be concomitantive to the concomitants out of which it is formed, aud of course also therefore to their common original.

Now the extension of this theorem, to which I wish to call attention, is this, that not only such determinant as a whole is a concomitant to such origin, but every minor system of determinants that can be formed out of it will form a concomitantive plexus complete within itself to the same original. But, much more generally, it should be observed that there is no occasion to begin with a square determinant; it is sufficient to have a rectangular array of terms formed by taking the several terms of one plexus or of several plexuses combined, provided that they are of the same degree in respect to the variables (or to the selected system of variables, if there be several systems), and forming out of such rectangular array any minor system of determinants at will. Every such system will be a concomitantive plexus. The simple illustrations which follow will make my meaning clear.

Suppose

$$\phi = ax^6 + 6bx^5y + 15cx^4y^2 + 21dx^3y^3 + 15ex^2y^4 + 6fxy^5 + gy^6.$$

I have previously remarked, in the foregoing sections, that a, b, c, d, e, f, g, the coefficients form an invariantive plexus to ϕ; so also we know that the catalecticant

$$\begin{matrix} a & b & c & d \\ b & c & d & e \\ c & d & e & f \\ d & e & f & g \end{matrix}$$

is an invariant to ϕ. But we are now able to couple together these facts and see the law which is contained between them;

for if we take

$$\left(\frac{d}{dx}\right)^{\iota}\phi,\ \left(\frac{d}{dx}\right)^{\iota-1}\cdot\frac{d}{dy}\,\phi\ \ldots\ .\ \left(\frac{d}{dy}\right)^{\iota}\phi,$$

ι being any number; as for instance, if we take $\iota = 3$, we shall have as a plexus

$$\begin{aligned} &ax^3 + 3bx^2y + 3cxy^2 + dy^3,\\ &bx^3 + 3cx^2y + 3dxy^2 + ey^3,\\ &cx^3 + 3dx^2y + 3exy^2 + fy^3,\\ &dx^3 + 3ex^2y + 3fxy^2 + gy^3; \end{aligned}$$

accordingly not only is the determinant

$$\begin{matrix} a & b & c & d\\ b & c & d & e\\ c & d & e & f\\ d & e & f & g \end{matrix}$$

an invariant, but also the system obtained by striking out any one line and one column, being what I term the first minors, will be an invariantive plexus: so too will the system of second minors

$$ac - b^2,\ bd - c^2,\ ce - d^2,\ ad - bc,\ ae - bd,\ be - cd,\ \&c.$$

form an invariantive plexus, as well as the last minors, *i.e.* the simple terms a, b, c, d, e, f, g. Again, we might have taken the plexus

$$\left(\frac{d}{dx}\right)^{2}\phi\ \ \frac{d}{dx}\cdot\frac{d}{dy}\phi\ \left(\frac{d}{dy}\right)^{2}\phi,$$

which would give the array

$$\begin{matrix} a & b & c & d & e\\ b & c & d & e & f\\ c & d & e & f & g; \end{matrix}$$

but the minor systems of determinants herein comprised will be found to be identical with those last considered, with the exception that the highest system, containing a single determinant only, will now be wanting. So in general it will easily be seen that a similar method in general, when ϕ is of 2ι dimensions, will lead to $\iota + 1$ invariantive plexuses comprising the given coefficients grouped together at one extremity of the scale, and the catalecticant alone at the other; and if ϕ is of $2\iota + 1$ dimensions, there will still be

$\iota + 1$ such plexuses, commencing with the coefficients as one group and ending with a system of combinations of the $(\iota+1)^{\text{th}}$ degree in regard of the coefficients, which system accordingly takes the place of the catalecticant of the former case, which for this case is nonexistent.

As a profitable example of the application of this law of synthesis, in its present extended form, let it be required to determine the conditions that a function of x, y of the fifth degree may have three equal roots. In general, let $\phi = ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + ey^5$, then ϕ has a quadratic and cubic covariant of which I have written at large in my supplemental essay above referred to, being in fact the s and t (*i.e.* the quadrinvariant and cubinvariant) in respect to x', y' (x, y being treated as constants) of

$$\left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^4 \phi.$$

Let these covariants respectively be called

$$Ax^2 + 2Bxy + Cy^2 = u,$$
$$ax^3 + 3\beta x^2y + 3\gamma xy^2 + \delta y^3 = v;$$

then
$$\left.\begin{matrix} Ax + By \\ Bx + Cy \end{matrix}\right\} \text{ forms a plexus,}$$

and
$$\left.\begin{matrix} ax^2 + 2\beta xy + \gamma yz \\ \beta x^2 + 2\gamma xy + \gamma yz \end{matrix}\right\} \text{ will form another.}$$

Now when $a = 0$, $b = 0$, $c = 0$, ϕ will have 3 equal roots, and $\left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^4 \phi$ becomes

$$6dy.x'^2y'^2 + 4(dx + ey)\,x'y'^3 + (ex + fy)\,y'^4,$$

the quadrinvariant in respect to x', y' of which is easily seen to be d^2y^2 and the cubicinvariant d^2y^3. Accordingly the grouping $\begin{matrix} A & B \\ B & C \end{matrix}$ becomes $\left.\begin{matrix} 0 & 0 \\ 0 & d^2 \end{matrix}\right\}$, and the grouping $\left.\begin{matrix} a & \beta & \gamma \\ \beta & \gamma & \delta \end{matrix}\right\}$ becomes $\left.\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & d^3 \end{matrix}\right\}$. Accordingly, we see that the determinant $\left.\begin{matrix} A & B \\ B & C \end{matrix}\right\}$ and all the first minors of $\left\{\begin{matrix} a & \beta & \gamma \\ \beta & \gamma & \delta \end{matrix}\right.$

i.e. $$a\gamma - \beta^2, \quad \beta\delta - \gamma^2, \quad a\delta - \beta\gamma,$$

become zero; but the former single quantity $\left.\begin{matrix} A & B \\ B & C \end{matrix}\right\}$ being an invariant, and this last system being an invariantive

plexus, all the quantities so affirmed to be zero will remain zero, notwithstanding any linear transformations to which ϕ may be subjected; thus then we obtain an immediate proof of the theorem that when a function of x and y of the fifth degree contains three equal roots the determinant of its quadratic covariant, which in fact is its sole quart-invariant, and the first minors of its cub-invariant will be all separately zero. This theorem may be made still more stringent, for by combining

$$Ax^2 + 2Bxy + Cy^2,$$
$$\alpha x^2 + 2\beta xy + \gamma y^2,$$
$$\beta x^2 + 2\gamma xy + \delta y^2,$$

it becomes manifest that in the case supposed all the first minor determinants of

$$\begin{matrix} A & B & C \\ \alpha & \beta & \gamma \\ \beta & \gamma & \delta \end{matrix}$$

will be zero, shewing in addition to the theorem as last enumerated that also

$$A : B : C :: \alpha : \beta : \gamma :: \beta : \gamma : \delta.$$

It is curious and instructive to remark that this last set of equations, stringent as *they appear*, and far more than enough to express a duplex condition, are not sufficient to imply unequivocally the existence of three equal roots, unless we have also $AC - B^2 = 0$; for suppose ϕ to take the form $ax^5 + fy^5$ (b, c, d, e all vanishing), then it will easily be seen that

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 0,$$
$$A = 0, \quad B = af, \quad C = 0.*$$

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* If we take L, M, N a system of fundamental invariants to ϕ, of which all the other invariants of ϕ are rational integer functions, $L = A\ B\ C$, and the simplest forms for M and N are $B\ C$

$$M = \begin{matrix} A & B & C \\ \alpha & \beta & \gamma \\ \beta & \gamma & \delta \end{matrix} \quad \text{and} \quad N = \begin{matrix} \alpha & 2\beta & \gamma & \\ & \alpha & 2\beta & \gamma \\ \beta & 2\gamma & \delta & \\ & \beta & 2\gamma & \delta \end{matrix}$$

where L and N are the discriminants of the quadratic and cubic covariants of ϕ respectively, and a linear function of M, L^2 is the discriminant of ϕ itself (L, M, N being of 4, 8, and 12 dimensions respectively in the coefficients of ϕ).

For many purposes of the calculus of forms it is desirable to have the command of cases for which any two out of these three invariants may be made to vanish without the third vanishing; and it will be found that when ϕ is of the form $y^3(cx^2+fy^2)$, $L=0$, $M=0$; when ϕ is of the form $y(bx^4+fy^4)$, $N=0$, $L=0$; and when ϕ is of the form ax^5+ey^5, $M=0$, $N=0$;

Consequently we shall still have all the first minors of

$$\left.\begin{matrix} A & B & C \\ \alpha & \beta & \gamma \\ \beta & \gamma & \delta \end{matrix}\right\},$$

although there is not even so much as a pair of equal roots in ϕ; $AC - B^2$ however, it will be observed, is not zero in this supposition.

The theory of Hessians, simple or bordered, may be regarded as one among the infinite diversity of applications of the principle of the plexus. Let U, V, W, &c. be any number of concomitants having the common system of variables $x, y \ldots z$. Let χ represent

$$x' \frac{d}{dx} + y' \frac{d}{dy} + \ldots + z \frac{d}{dz},$$

and take $\quad \chi^2 U + \lambda\chi V + \&c. + \mu\chi W = S,$

then $\quad \frac{dS}{dx'}, \frac{dS}{dy'}, \ldots \frac{dS}{dz'}$ forms a plexus;

and this, combined with χV, &c. $\ldots \chi W$, enables us to eliminate dialytically $x', y', z', \lambda \ldots \mu$. The result is a Hessian of U, bordered with

$$\frac{dV}{dx}, \frac{dV}{dy}, \ldots \frac{dV}{dz}$$

horizontally and vertically, and also with

$$\begin{matrix} \frac{dW}{dx}, & \frac{dW}{dy}, \ldots & \frac{dW}{dz} \\ \&c. & \&c. & \end{matrix}$$

similarly dispersed; which Hessian, so bordered, is thus seen to be a concomitant to $U, V, \ldots W$. The Hessian, as ordinarily bordered with $\xi, \eta, \ldots \zeta$, is derived by taking for V the universal concomitant $x\xi + y\eta + \ldots + z\zeta$, and for W (if there be a double border)

$$x\xi' + y\eta' + \ldots + z\zeta',$$

and so forth.

If V be taken identical with U, the resulting form, con-

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and of course when ϕ is of the form $y^3(dx^2 + fy^2)$, $L=0$, $M=0$, $N=0$; it being obviously true in general, as remarked by Mr. Cayley, that when not less than half the roots of a function of two variables are equal, all its invariants must vanish together.

sisting of U bordered with $\frac{dU}{dx}$, $\frac{dU}{dy}$,... $\frac{dU}{dz}$, has been shewn in my paper 'On certain general Properties of Homogeneous Functions,' in this *Journal*, to be equal the product of the simple Hessian of U and of U itself multiplied by a numerical factor. The theory of the bordered Hessian may be profitably extended by taking

$$S = \chi^{2r}.U + \lambda\chi^{r}.V + \ldots + \mu\chi^{r}.W,$$

and combining with $\chi^{r}.V \ldots\ldots \chi^{r}.W$, the plexus obtained by operating upon S with the r^{e} powers and products of $\frac{d}{dx}$, $\frac{d}{dy}$,... $\frac{d}{dz}$, and eliminating dialytically the r^{e} powers and products of $x', y' \ldots z'$. Thus if

$$U = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4 \text{ and } V = (x\xi + y\eta)^2,$$

we obtain, by taking $S = \chi^4.U + \lambda\chi^3.V$, and proceeding as indicated in the preceding,

$$\begin{array}{cccc} a & b & c & \xi^2 \\ b & c & d & \xi\eta \\ c & d & e & \eta^2 \\ \xi^2 & \xi\eta & \eta^2 & \end{array}$$

as a concomitant to U. So again, if

$$U = ax^5 + 5bx^4y + \&c. + fy^5,$$

we find

$$\begin{array}{cccc} ax + by & bx + cy & cx + dy & \xi^2 \\ bx + cy & cx + dy & dx + ey & \xi\eta \\ cx + dy & dx + ey & ex + fy & \eta^2 \\ \xi^2 & \xi\eta & \eta^2 & \end{array}$$

a concomitant to U.

These extensions of the ordinary theory of Hessians will be found to be of considerable practical importance in the treatment of forms, for which reason they are here introduced.

Section VI.

On the partial Differential Equations to Concomitants, Orthogonal and Plagiogonal Invariants, &c.

In the 7th note of the Appendix to the three preceding sections I alluded to the partial differential equations by which every invariant may be defined.

This method may also be extended to concomitants generally. M. Aronhold, as I collect from private information, was the first to think of the application of this method to the subject; but it was Mr. Cayley who communicated to me the equations which define the invariants of functions of two variables.* The method by which I obtain these equations and prove their sufficiency is my own, but I believe has been adopted by Mr. Cayley in a memoir about to appear in *Crelle's Journal.* I have also recently been informed of a paper about to appear in *Liouville's Journal* from the pen of M. Eisenstein, where it appears the same idea and mode of treatment have been made use of. Mr. Cayley's communication to me was made in the early part of December last, and my method (the result of a remark made long before) of obtaining these and the more general equations, and of demonstrating their sufficiency, imparted a few weeks subsequently—I believe between January and February of the present year.

The method which I employ, in fact, springs from the very conception of what an invariant means, and does but throw this conception into a concise analytical form.

Suppose, to fix the ideas,

$$\phi = ax^n + nbx^{n-1}.y + n.\tfrac{1}{2}(n-1).cx^{n-2}.y^2 + \&c. \ldots + ly^n,$$

and let $I(a, b, c \ldots l)$ be any invariant to ϕ.

Now suppose x to become $x + ey$, but y to remain unchanged, the modulus of the transformation $\begin{matrix} 1 & e \\ 0 & 1 \end{matrix}\Big\}$ being unity, I cannot alter in consequence of this substitution; but the effect of this substitution is to convert ϕ into the form

$$\alpha x^n + n\beta x^{n-1}.y + n.\tfrac{1}{2}(n-1).\gamma x^{n-2}.y^2 + \&c. + \lambda y^n,$$

where $\alpha = a$, $\beta = b + ae$, $\gamma = c + 2be + ae^2$, &c. &c.

$$\lambda = l + \&c. \&c. + nbe^{n-1} + ae^n.$$

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* It is extremely desirable to know whether M. Aronhold's equations are the same in form as those here subjoined. It is difficult to imagine what else they can be in substance. Should these pages meet the eye of that distinguished mathematician he will confer a great obligation on the author and be rendering a service to the theory by communicating with him on the subject: and I take this opportunity of adding that I shall feel grateful for the communication of any ideas or suggestions relating to this new Calculus from any quarter and in any of the ordinary mediums of language—French, Italian, Latin, or German, provided that it be in the Latin character. My address in London will be found appended at the end of this paper.

Consequently, if we make

$$\Delta b = ae, \quad \Delta c = 2bc + ae^2, \text{ \&c. \&c.},$$

we have by Taylor's theorem, observing that $\Delta a = 0$,

$$\Delta I = (\Delta b . \frac{d}{db} + \Delta c . \frac{d}{dc} \text{ \&c.}) I + \frac{1}{1.2} (\Delta b . \frac{d}{db} + \Delta c \frac{d}{dc} \text{ \&c.})^2 I$$

$$+ \frac{1}{1.2\,3} (\Delta b \frac{d}{db} + \text{\&c.})^3 I, \text{ \&c. \&c.} = 0;$$

and this being true for all the values of e, every separate coefficient of e in ΔI must be zero: hence we obtain n different equations by equating to zero the coefficients of e, e^2, ... e^n respectively. The first of these equations will be

$$\left(a \frac{d}{db} + 2b \frac{d}{dc} + 3e \frac{d}{de} + \text{\&c.} \right) \phi = 0,$$

and it is obvious that this will imply all the rest; for, when e is taken indefinitely small, $I(a, b, c \ldots)$ does not alter (when this equation is satisfied) by changing $a, b, c \ldots$ into $a', b', c' \ldots$; consequently $I(a', b', c', \text{\&c.})$ will not alter, when in place of a', b', c' we write a'', b'', c'', &c., obtained from a', b', c', &c., by the same law as a', b', c', &c., from a, b, c, &c.

Thus we may go on giving an indefinite number of increments, ey to x, without changing the value of I. Consequently, if the equation above written be satisfied, *à priori* all the rest must be so too. But there is not any difficulty in shewing the same thing by a direct method.*

For we have

$$\left(a \frac{d}{db} + 2b \frac{d}{dc} + 3c \frac{d}{dd} + \text{\&c.} \right) I = 0,$$

an identical equation. Hence

$$\left(a \frac{d}{db} + 2b \frac{d}{dc} + 3c \frac{d}{dd} \text{ \&c.} \right) \left\{ \left(a \frac{d}{db} + 2b \frac{d}{dc} + 3c \frac{d}{dd} \text{ \&c.} \right) I \right\} = 0,$$

* The method above given has the advantage however of being immediately applicable to every species of concomitant, and we learn from it that concomitance, whether absolute or conditional, is sufficiently determined when affirmed to exist for *infinitessimal* variations; it cannot exist for infinitessimal variations without, by *necessary implication*, existing for finite variations also; a most important consideration this in conducing to a true idea of the nature of invariance and the other kinds of concomitance, and in cutting off all superfluous matter from the statement of the conditions by which they are defined.

hence

$$\left\{\left(a\frac{d}{db}+2b\frac{d}{dc}+3c\frac{d}{dd}\,\&c.\right)\right\}\left\{\left(a\frac{d}{db}+2b\frac{d}{dc}+3c\frac{d}{dd}\,\&c.\right)\right\}I$$
$$+\left\{a\frac{d}{db}+2b\frac{d}{dc}+3c\frac{d}{dd}\,\&c.\right\}^2 I=0;$$

i. e.
$$\left\{2\left(a\frac{d}{dc}+3b\frac{d}{dd}+6c\frac{d}{de}\,\&c.\right)\right.$$
$$\left.+\left(a\frac{d}{db}+2b\frac{d}{dc}+3c\frac{d}{dd}\,\&c.\right)^2\right\}I=0:$$

repeating the application of the symbolic operator

$$\left(a\frac{d}{db}+2b\frac{d}{dc}\,\&c.\right),$$

we obtain

$$\left.\begin{aligned}&1.2.3\left\{a\frac{d}{dd}+4b\frac{d}{de}+10c\frac{d}{df}\,\&c.\right\}\\&+1.2\left\{a\frac{d}{db}+2b\frac{d}{dc}+\&c.\right\}.\left\{a\frac{d}{dc}+3b\frac{d}{dd}\,\&c.\right\}\\&+\left(a\frac{d}{db}+2b\frac{d}{dc}+3c\frac{d}{dd}\,\&c.\right)^3\end{aligned}\right\}I=0,$$

and so on; the numerical multipliers of the terms of the several series within the parentheses forming the regular succession of figurate numbers 1, 2, 3, &c.
1, 3, 6, &c.
1, 4, 10, &c.

It is easy to see that these equations correspond to the results of making the coefficients of the successive powers of e equal to zero.

I may remark, that the first instance as far as I know on record of this, (as some may regard it rather bold) but in point of fact perfectly safe and legitimate method of differentiating conjointly operator and operand, occurs in a paper by myself in this *Journal*, Feb. 1851, "On certain general properties of homogeneous functions;" where I have applied it in operating with

$$\left\{(x_1-a_1e)\frac{d}{da_1}+(x_2-a_2e)\frac{d}{da_2}\,\&c.\right\}$$

upon
$$\left\{(x_1-a_1e)\frac{d}{da_1}+(x_2-a_2e)\frac{d}{da_2}\,\&c.\right\}^r.\omega;$$

which, as I have there noticed, gives the result

$$\left\{(x_1 - a_1 e)\frac{d}{da_1} + \&c.\right\}^{r+1}.\omega$$

$$- re\left\{(x_1 - a_1 e)\frac{d}{da_1} + \&c.\right\}^{r}.\omega.$$

The equation $\left(a\frac{d}{db} + 2b\frac{d}{de} + \&c.\right)I = 0$ is evidently not enough to define I as an invariant; it merely serves to shew that I does not alter when in place of x we write $x + ey$, but this is true for any function of the differences of the roots of $\left(\frac{x}{y}\right)$ multiplied by a suitable power of a, viz. that power which is just sufficient to cause the product to become integer. But if we now, for convenience, write

$$\phi = ax^n + n.bx^{n-1}.y + n.\tfrac{1}{2}(n-1)\,cx^{n-2}.y^2 + \&c. \ldots .$$
$$+ n.\tfrac{1}{2}(n-1)\,c^1.x^2y^{n-2} + x.b^1ay^{n-1} + a^1y^n,$$

and form the similar equation from the other side, viz.

$$\left(a'\frac{d}{db'} + 2b'\frac{d}{dc'} + 3c'\frac{d}{dd'} + \&c.\right)I = 0,$$

these two equations together will suffice to define any invariant, as I shall proceed to shew—these are the two equations alluded to brought under my notice by Mr. Cayley. If they coexist, it follows from the method by which I have deduced them that x may be changed into $x + ey$, or y into $y + fx$, without I being altered, e and f having any values whatever: and it is obvious that these substitutions may be performed, not merely alternatively but successively, because the equations between the coefficients are identical equations, and depend only on the form of I.

Let now x become $x + ey$, and then y become $y + fx$; the result of these substitutions is to convert

$$x \text{ into } x + efx + ey,$$

and

$$y \text{ into } fx + y.$$

Finally, let x become $x + gy$; then x is converted into $(1 + ef)(x + gy) + ey$, and y into $y + f(x + gy)$,

i.e. x becomes $(1 + ef)x + (eg + efg)y$,

and y becomes $fx + (1 + fg)y$.

The modulus of substitution it is evident, *a priori*, always remains unity, and nothing would be gained by pushing the

substitutions any further, as it is clear that we may satisfy the equations

$$1 + ef = a, \quad e + g + efg = b,$$
$$f = a', \qquad 1 + fg = b',$$

for all values of a, b, a', b', which satisfy the equation

$$ab' - a'b = 1;$$

and for none other except such values; hence I remains unaltered for any unit-modular linear transformation of x, y, and is therefore an invariant by definition.

If ϕ be taken a function of three variables, x, y, z, and be thrown under the form

$$az^n + (a_1x + b_1y)z^{n-1} + (a_2x^2 + 2b_2xy + c_2y^2)z^{n-2} + \&c.,$$

and z be any invariant of ϕ, by supposing x to become $x + ey$, and giving b_1, b_2, c_2, &c., the corresponding variations, and taking e indefinitely small, we obtain

$$\left\{a_1\frac{d}{db_1} + \left(a_2\frac{d}{db_2} + 2b_2\frac{d}{dc_2}\right) + \left(a_3\frac{d}{db_3} + 2b_3\frac{d}{dc_3} + 3c_3\frac{d}{dd_3}\right) + \&c.\right\} I = 0,$$

$$\left\{b_1\frac{d}{da_1} + \left(c_2\frac{d}{db_2} + 2b_2\frac{d}{da_2}\right) + \&c.\ \&c.\right\} I = 0:$$

and in like manner, by arranging ϕ according to the powers of y and of x, we obtain two other pairs of equations: it is clear, however, that three equations (it would seem any three out of the six) would suffice and imply the other three. The method of demonstration will be the same as in the instance of two variables: 1st, It can be shewn by the method of successive accretions, that I remaining invariable when x receives an indefinitely small increment ϵy, or z an indefinitely small increment ϵx, or z an indefinitely small increment ϵx, it will also remain invariable when these increments are taken of any finite magnitude. 2nd, By eight successive transformations, admissible by virtue of the preceding conclusion, x, y, z may be changed into any linear functions of x, y, z, consistent with the modulus of transformation being unity. And in general for a function of m variables, m partial differential equations similarly constructed (but not however arbitrarily selected) will be necessary and sufficient to determine any invariant: and it is clear that all the general

properties of invariants must be contained in and be capable of being educed out of such equations.

The same method enables us also to establish the partial differential equations for any covariant, or indeed any concomitant whatever.

Thus let

$$\phi = ax^n + nbx^{n-1}y + n\tfrac{1}{2}(n-1)\,cx^{n-2}y^2 + \&c. + nb'xy^{n-1} + a'y^n = 0,$$

and let $K(a, b, c, \&c.;\ x, y,\ x', y', \&c.;\ \xi, \eta, \&c.)$ represent any concomitant, x, y; x', y' being cogredient, and ξ, η, &c. contragredient systems; when x, y become $x + ey,\ y$, any such systems x', y' becomes $x' + ey', y'$; and any such system as ξ, η become $\xi, \eta - e\xi$; and taking e indefinitely small, the second coefficients a, b, c, &c. become $a, b + ae,\ c + 2be$, &c. as before; hence the equation to the concomitant becomes

$$\left\{a\frac{d}{db} + 2b\frac{d}{dc} + \&c. - y\frac{d}{dx} - y'\frac{d}{dx'} + \&c. + \xi\frac{d}{d\eta} - \&c.\right\} = 0;^*$$

and in like manner, by changing y into $y + ex$, results the corresponding equation

$$\left\{a'\frac{d}{db'} + 2b'\frac{d}{dc'} + \&c. - x\frac{d}{dy} - x'\frac{d}{dy} + \&c. + \eta\frac{d}{d\xi}\ \&c.\right\} K = 0.$$

These two equations define in a perfectly general manner every concomitant (with any given number of cogredient and contragredient systems) to the form ϕ; and the due number of pairs of similarly constituted equations will serve to define the concomitant to a function of any given number of variables.†

In like manner we may proceed to form the equations corresponding to what may be termed *conditional* concomitants, whether *orthogonal* or *plagiogonal*. The concomitants previously considered may be termed absolute, the linear transformations admissible being independent of any but the one general relation, imposed merely for the purpose of convenience, viz. of their modulus being made unity. An orthogonal concomitant is a form which remains invariable, not for arbitrary unit-modular, but for orthogonal transformation, *i. e.* for linear substitutions of $x, y \ldots : z$, which leave unchanged $x^2 + y^2 + \ldots + z^2$: in like manner, a plagiogonal

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* For we have

$$K(a, b+ab, c+ac, \&c.;\ x, y, \&c.;\ \xi, \eta, \&c.),$$
$$= K(a, b, c, \&c.;\ x, x+ey, \&c.;\ \xi, \eta-e\xi, \&c.;\ \&c.)$$

† Vide Note (10).

concomitant may be defined of a form which remains invariable for all linear substitutions of $x, y \ldots z$, which leave unaltered any given quadratic function of $x, y \ldots z$. Thus, let it be required to express the condition of $Q(a, b, c \ldots x, y\,;\, \xi, \eta)$, being a conditional concomitant to the form

$$ax^n + nbx^{n-1}y + \&c. + nb'xy^{n-1} + a'y^n.$$

Let x become $x + ey$, e being indefinitely small, then y must become $y - ex$, and the variations of $a, b, \ldots b', a'$ will be the sum of the variations produced by taking separately $x + ey$ for x and $y - ex$ for y. Hence the one sole condition for Q being of the required form becomes

$$\left\{\begin{aligned} &a\frac{d}{db} + 2b\frac{d}{dc} + \&c. - y\frac{d}{dx} + \xi\frac{d}{d\eta} \\ &a'\frac{d}{db'} + 2b'\frac{d}{dc'} + \&c. - x\frac{d}{dy} + \eta\frac{d}{d\xi} \end{aligned}\right\} Q = 0\,;$$

or, as it may be written, $\theta.Q - \omega.Q = 0$ where $\theta.Q = 0$, $\omega.Q = 0$ are the two equations expressing the conditions of Q, being an unconditional or absolute concomitant; and so in general if ϕ be a function of m variables, we may obtain $\frac{1}{2}m(m-1)$ equations of the form $L - M = 0$ for the concomitant, of which however $(m-1)$ only will be independent.

Supposing, again, the substitutions to which x, y are subject to be conditioned by $lx^2 + 2mxy + xy^2$ remaining unalterable, or which is a more convenient and only in appearance less general supposition by $x^2 + 2mxy + xy^2$ remaining unalterable, the general type of an infinitesimal system of substitutions will be rendered by supposing x, y to become $(1 + me)\,x + ey$, $-ex + (1 - me)\,y$, respectively, for then $x^2 + 2mxy + y^2$ becomes

$$(1 - m^2e^2)\,x^2 + \{2m + (2m - 2m^2)\,e^2\}\,xy + (1 - m^2e^2)\,y^2,$$

which differs from $x^2 + 2mxy + y^2$ only by quantities of the second order of smallness which may be neglected, and ξ and η will therefore become $(1 - me)\,\xi - e\eta$, $-ex + (1 + me)\,y$, respectively: then, as to the coefficients of ϕ in addition to the variations which they undergo when m is zero, there will be the variations consequent upon x, assuming the increment mex and y the increment $-mey$: but by making x become $x + mex$, a, b, c, &c. b', a' assume respectively the variations

$$n.mea,\ (n-1)\,meb,\ \&c.\ meb',\ 0,\ \text{respectively};$$

and by making y become $y - mey$, the corresponding variations become

$$0, \; -meb, \; \ldots \; -(n-1)\, meb', \; na', \text{ respectively.}$$

Hence the equation becomes

$$\theta.Q - \omega Q + m(\lambda Q - \mu Q) = 0,$$

where θ and ω have the same signification as before, and where λ denotes

$$na\frac{d}{da} + (n-1)\, b\frac{d}{db} + \ldots\ldots + b'\frac{d}{db'} + x\frac{d}{dx} - \xi\frac{d}{d\xi},$$

and μ denotes

$$b\frac{d}{db} + 2c\frac{d}{dc} + \ldots\ldots\ldots + na'\frac{d}{da'} - y\frac{d}{dy} + \eta\frac{d}{d\eta}.$$

If there be several systems of x, y or of ξ, η, or of both, the only difference in the equation of condition will consist in putting

$$\Sigma\left(y\frac{d}{dx}\right), \quad \Sigma\left(x\frac{d}{dy}\right), \quad \Sigma\left(x\frac{d}{dx}\right), \quad \Sigma\left(y\frac{d}{dy}\right),$$

$$\Sigma\left(\eta\frac{d}{d\xi}\right), \quad \Sigma\left(\xi\frac{d}{d\eta}\right), \quad \Sigma\left(\xi\frac{d}{d\xi}\right), \quad \Sigma\left(\eta\frac{d}{d\eta}\right),$$

instead of the single quantities included within the sign of definite summation.

Fearing to encroach too much on the limited space of the *Journal*, I must conclude for the present with shewing how to integrate the general equation to the orthogonal invariant of ϕ the general function of x, y.

Beginning with $\phi = ax^2 + 2bxy + cy^2$, the equation becomes

$$\left\{-2b\frac{d}{da} + (a-c)\frac{d}{db} + 2b\frac{d}{dc} + y\frac{d}{dx} - x\frac{d}{dy}\right\} Q = 0.$$

Write now $\quad da = -2bd\theta, \quad dx = yd\theta,$

$$db = (a-c)\, d\theta, \quad dy = -xd\theta,$$

$$dc = +2bd\theta;$$

we have then

$$\lambda da + \mu db + \nu dc = d\theta\,\{\mu a + 2(\nu - \lambda)\, b - \mu c\}.$$

Let $\quad \mu = \kappa\lambda; \quad 2(\nu - \lambda) = \kappa\mu; \quad -\mu = \kappa\nu;$

then $\quad d\log(\lambda a + \mu b + \nu c) = \kappa d\theta;$

or $\quad \lambda a + \mu b + \nu c = be^{\kappa\theta}.$

To find κ we have the determinant

$$\left.\begin{matrix} \kappa & -1 & 0 \\ 2 & \kappa & -2 \\ 0 & 1 & \kappa \end{matrix}\right\} = 0,$$

that is,

$$\kappa^3 + 4\kappa = 0,$$

and calling the 3 roots of this equation κ_1, κ_2, κ_3, we have

$$\kappa_1 = 0, \quad \kappa_2 = 2\iota, \quad \kappa_3 = -2\iota;$$

accordingly we may put

$$\kappa = 0, \quad \lambda = 1, \quad \mu = 0, \quad \nu = 1,$$

or

$$\kappa = 2\iota, \quad \lambda = 1, \quad \mu = 2\iota, \quad \nu = -1,$$

or

$$\kappa = -2\iota, \quad \lambda = 1, \quad \mu = -2\iota, \quad \nu = -1.$$

Again,

$$pdx + qdy = (py - qx)\, d\theta;$$

and putting $-q = e.p$, $p = e.q$, so that $px + qy = E.e^{\iota\theta}$,

$$e^2 = -1, \quad e_1 = \iota, \quad e_2 = -\iota;$$

and we may put

$$e = \iota, \quad p = 1, \quad q = -\iota,$$

or

$$e = -\iota, \quad p = 1, \quad q = +\iota.$$

Consequently the complete integral of the given partial differential equation is found by writing

$$a + c = l, \qquad x - \iota y = E.e^{2\theta},$$
$$a + 2\iota b - c = l' e^{2\iota\theta}, \qquad x + \iota y = E'.e^{-\iota\theta},$$
$$a - 2\iota b - c = l' e^{-2\iota\theta}.$$

By means of these 5 equations, after eliminating θ, we may obtain 4 independent equations between a, b, c; x, y. Suppose

$$Q_1 = 0, \quad Q_2 = 0, \quad Q_3 = 0, \quad Q_4 = 0;$$

then $Q = F(Q_1, Q_2, Q_3, Q_4)$ is the complete integral required.

Pursuing precisely the same method for the general case, it will be found that, calling the degree of the given function n when n is even, the equation in κ to be solved will be

$$\kappa(\kappa^2 + 4)(\kappa^2 + 9) \ldots (\kappa^2 + n^2) = 0;$$

and when n is odd (say $2m + 1$), the equation in κ to solve will be

$$(\kappa + 1)(\kappa^2 + 9) \ldots (\kappa^2 + n^2) = 0;$$

and performing the necessary reductions, and calling the roots of the equation, arranged in order of magnitude, $\kappa_1{}^{\iota}$, $\kappa_2{}^{\iota} \ldots \kappa_n{}^{\iota}$, respectively, it will be found that the equations

containing the integral become

$$\left.\begin{aligned} L_1 &= l_1 e^{\kappa_1 \iota\theta} \\ L_2 &= l_2 e^{\kappa_2 \iota\theta} \\ L_3 &= l_3 e^{\kappa_3 \iota\theta} \\ &\&c. \quad \&c. \\ L_{n+1} &= l_{n+1} e^{\kappa_{n-1} \iota\theta} \end{aligned}\right\} \quad \left.\begin{aligned} x - \iota y &= E e^{\iota\theta} \\ x + \iota y &= E' e^{-\iota\theta} \end{aligned}\right\}$$

where $l_1, l_2 \ldots l_{n+1}$; E, E' are arbitrary constants, and where $L_1, L_2 \ldots L_{n+1}$ are the values assumed by the 1st, 2nd ... $(n+1)^{\text{th}}$ coefficients of the given function ϕ, or

$$ax + nbx^{n-1} + \&c. + nb'xy^{n-1} + by^n,$$

when it is transformed by writing $x + \iota y$ in place of x, and $y + \iota x$ in place of y. ι is of course employed in the foregoing according to the usual annotation to represent $\sqrt{-1}$. The same method applies to the general theory of plagiogonal concomitants, where the linear substitutions are supposed such as to leave $lx^2 + 2mxy + xy^2$ unaltered in form, and the equations in θ which contain the integral present themselves under a similar aspect. But a more full discussion of these interesting integrals must be reserved until the ensuing number of the *Journal*.

[*To be continued.*]

26, *Lincoln's Inn Fields,*
April 1, 1852.

NOTES IN APPENDIX.

(9) The scale of covariants to a function of (x, y) obtained by the method of unravelment (*Journal*, No. XXVIII. p. 66) may be otherwise deduced in a form more closely analogous to that in which the corresponding theorems for the corresponding invariantive scale (No. XXVIII. p. 65) by a method which has the advantage of exhibiting the scale equally well for the case of functions of the degree $4\iota + 2$ or $4\iota + 4$, the only difference being that in the latter case the coefficients of the odd powers of λ will be found all to vanish, so that the degrees of the covariants will rise by steps of 4 instead of by steps of 2, just conversely to what happens in the invariantive scale; whereas in the invariantive scale alluded to the forms containing odd powers of λ vanish when the degree of the function is of the form $4\iota + 2$, but do not vanish when it is of the form 4ι. This method in the form here subjoined is a slight modification of one suggested to me by my friend Mr. Cayley.

Let F be the given function of x, y of the degree $2n$; take the systems x', y'; x_1, y_1 cogredient with one another and with x, y. Then form the concomitant

$$K = \left(x' \frac{d}{dx} + y' \frac{d}{dy}\right)^n F + \lambda (x'y - y'x)^{n-1} (x'y_1 - y'x_1) (xy_1 - yx_1).$$

Then (by what may be termed the Divellent method, and which has been previously applied by me in the *Philosophical Magazine* for Nov. 1851) calling $\theta_0, \theta_1, \theta_2 \ldots \theta_n$, the coefficients of

$$x^n, x^{n-1}, y', \ldots y^n \text{ in } K,$$

we shall have

$$\theta_0 = A_0x^n + B_0x^{n-1}y + \ldots L_0y^n,$$
$$\theta_1 = A_1x^n + B_1x^{n-1}y + \ldots L_1y^n,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\theta_n = A_nx^n + B_nx^{n-1}y + \ldots L_ny^n,$$

the coefficients being functions of the coefficients of f and of quadratic combinations of x_1, y_1, affected with the multiplier λ, and the determinant

$$A_0, B_0 \ldots L_0,$$
$$A_1, B_1 \ldots L_1,$$
$$\ldots\ldots\ldots$$
$$\ldots\ldots\ldots$$
$$A_n, B_n \ldots L_n,$$

will give a function of λ in which the coefficients of the several powers of λ will be all zero or covariants of F.

The actual form of this determinant is not here given for want of space and time, but will be exhibited hereafter. Precisely an analogous method applies to obtain the scale to $(x, y, z)^4$ given in Note (2). Calling $F = (x, y, z)^4$, let the systems x', y', z'; x_1, y_1, z_1, be taken cogredient with one another and with x, y, z. Then, using R to express the determinant

$$x', y', z',$$
$$x, y, z,$$
$$x_1, y_1, z_1,$$

and making

$$K = \left(x'\frac{d}{dx} + y'\frac{d}{dy} + z'\frac{d}{dz}\right)^2 F + \lambda R,$$

and proceeding as above by the divellent method, we obtain the scale required.

(10). It is obvious that these defining equations ought to give the means of discovering and verifying all the properties of concomitants; but it is very difficult to see how in the present state of analysis many of the general theorems that have been stated, readily admit of being deduced from them.

The comparatively simple but eminently important theory of the evector symbol does however admit of a very pretty verification by aid of these equations. Thus, suppose θ any concomitant; suppose a contravariant to a function F of x, y, say

$$ax^n + nbx^{n-1}y + \ldots + nb'xy^{n-1} + a'y^n.$$

Then θ must satisfy the two equations

$$\left(L + \xi\frac{d}{d\eta}\right)\theta = 0, \quad \left(L' + \eta\frac{d}{d\xi}\right)\theta = 0,$$

where

$$L = a\frac{d}{db} + 2b\frac{d}{dc} + \ldots + nb'\frac{d}{da'},$$

$$L' = a'\frac{d}{db'} + 2b'\frac{d}{dc'} + \ldots + nb\frac{d}{da}.$$

Now let $\phi = \chi(\theta)$ where

$$\chi = \xi^n \frac{d}{da} + \xi^{n-1}\eta \frac{d}{db} + \xi^{n-2}\eta^2 \frac{d}{dc} + \ldots + \eta^n \frac{d}{da'};$$

then $$L(\chi\theta) = \chi(L\theta) - (\chi L)\theta,$$

$$= \chi(L\theta) - \left(\xi^n \frac{d}{db} + 2\xi^{n-1}\eta \frac{d}{dc} + \ldots + n\xi\eta^{n-1} \frac{d}{da'}\right)\theta,$$

$$\xi \frac{d}{d\eta}(\chi\theta) = \chi\left(\xi \frac{d}{d\eta}\theta\right) + \left(\xi \frac{d}{d\eta}\chi\right)\theta,$$

$$= \chi\left(\xi \frac{d}{d\eta}\theta\right) + \left(\xi^n \frac{d}{db} + 2\xi^{n-1}\eta \frac{d}{dc} + \ldots + n\xi\eta^{n-1} \frac{d}{da'}\right)\theta.$$

Hence $$\left(L + \xi \frac{d}{d\eta}\right)\chi(\theta) = \chi\left\{\left(L + \xi \frac{d}{d\eta}\right)\theta\right\} = \chi 0 = 0.$$

Similarly $$\left(L' + \eta \frac{d}{d\xi}\right)\chi\theta = 0.$$

Hence if θ is an integral of the two conditioning equations, so also is $\chi(\theta)$. In like manner, if θ be a covariant or any other kind of concomitant of F, it may be proved that its evectant $\chi(\theta)$ is the same.

(11). Very much akin with the supposed equations (p. 182) is the following most remarkable equation, which can be proved to exist. Let ϕ be a function of x and y of the 5th degree. Let P and Q be the quadratic and cubic covariants of ϕ. [P is of two dimensions in the coefficients and also in the variables, and Q of three dimensions in both. They are in fact the s and t (in respect to x' and y') of $\left(x' \frac{d}{dx} + y' \frac{d}{dy}\right)^4 \phi$]; then, giving P and Q proper numerical factors, it will be found that

$$H_4\phi + PH\phi + Q\phi = 0.$$

I believe that a similar equation connects any function of x and y above the 3rd degree with its first and second Hessians. The proof will be given in a subsequent section, where also I shall give a complete proof, which occurred to me immediately after sending the preceding note to the press, of the complete Theory of the Respondent by means of the general equations of concomitance.

POSTSCRIPT.

Since the preceding was in type, I have ascertained the existence and sufficiency of a general method for forming the polar reciprocal and probably also the discriminant to functions of any degree of three variables by an explicit process of permutation and differentiation. In particular I am enabled to give the actual rule for constructing the polar reciprocal and the discriminant curves of the 4th and 5th degrees. So far as regards the polar reciprocal of curves of the 4th degree Mr. Hesse has already given a method of obtaining it, but mine is entirely unlike to this, and rests upon certain extremely simple and universal principles of the calculus of

forms. The only thing necessary to be done in order to carry on the process to curves of the 6th or higher degrees, is to ascertain the relation of the discriminants of functions of 2 variables of those respective degrees to such of the fundamental invariants as are of an inferior order to the discriminant.

The theory applies equally well to surfaces and to functions of any number of variables, and may, I believe, without any serious difficulty be extended so as to reduce to an explicit process the general problem of effecting the elimination between functions of any degree and of any number of variables. The method above adverted to will appear in a subsequent Section.

[*To be continued.*]

ON THE LAWS OF ELASTICITY.

By William John Macquorn Rankine, C.E., F.R.S.E., F.R.S.S.A., etc.

Sect. VI.—*On the Application of the Method of Virtual Velocities to the Theory of Elasticity.*

25. Lagrange's method of Virtual Velocities having been applied to the problems of the equilibrium and motion of elastic media by Mr. Green (*Camb. Trans.*, VII.), and by Mr. Haughton (*Trans. Royal Irish Acad.*, XXI., XXII.), it is my purpose in this and the following Section to point out the mutual correspondence between the coefficients in the formulæ arrived at by these gentlemen, and the coefficients of elasticity which form the subject of the previous portion of this paper, and also to shew how far the laws of relation between the nine coefficients of elasticity of a homogeneous body, which I originally proved by a method chiefly geometrical, are capable of being deduced symbolically from equations found according to Lagrange's method.

26. The principle of Virtual Velocities, as applied to molecular action, is as follows. Let X, Y, Z, denote the total accelerative forces applied to any particle whose mass is m, of an elastic medium, through agencies distinct from molecular action (such as the attraction of gravitation): let u, v, w be the components of the velocity of m; let δx, δy, δz denote indefinitely small virtual variations of x, y, z; let S be the total accelerative molecular force applied to m, δs an indefinitely small virtual variation of the line along which

it acts; then the following equation

$$\Sigma\left[m\left\{\left(X-\frac{du}{dt}\right)\delta x+\left(Y-\frac{dv}{dt}\right)\delta y+\left(Z-\frac{dw}{dt}\right)\delta z\right\}\right]$$
$$+\Sigma(mS\delta s)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16),$$

(the summation Σ being extended to all the particles of the medium), expresses at once all the conditions of equilibrium and motion of every particle of the medium.

27. In applying this principle to the theory of elastic media, both Mr. Green and Mr. Haughton assume the following postulates:

First, that in calculation we may treat each particle m as if it were a small rectangular space $dx\,dy\,dz$, filled with matter of a certain density ρ: so that for the symbol Σm we may substitute that of a triple integration

$$\iiint\rho\,dx\,dy\,dz.$$

Secondly, that the virtual moment $mS\delta s$ of the total molecular force acting on any particle m, is capable of being expressed by the product of the small rectangular space $dx\,dy\,dz$ into the variation, δV, of a certain function V of the relative position of m and the other particles of the body.*

Equation 16 is thus transformed into the following:

$$\iiint\rho\left\{\left(X-\frac{du}{dt}\right)\delta x+\left(Y-\frac{dv}{dt}\right)\delta y+\left(Z-\frac{dw}{dt}\right)\delta z\right\}dx\,dy\,dz$$
$$+\iiint\delta V\,dx\,dy\,dz=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17).$$

28. The term *elasticity* properly comprehends those molecular forces only whose variations are produced by, and tend to produce, variations in the volume and figure of bodies. There are, therefore, conceivable kinds of molecular force, which are not included in the term *elasticity*. For example, let us take the forces which Mr. MacCullagh ascribed to the particles of the medium which transmits light.

Let ξ, η, ζ denote displacements of a point in the medium, parallel respectively to x, y, z. Then Mr. MacCullagh supposes the molecular forces to be functions of

$$\frac{d\eta}{dz}-\frac{d\zeta}{dy};\quad \frac{d\zeta}{dx}-\frac{d\xi}{dz};\quad \frac{d\xi}{dy}-\frac{d\eta}{dx};$$

* This amounts in fact to the assumption that no part of the power developed by a variation of the relative positions of the particles is permanently converted into heat, or any other agency: in other words, that the body is perfectly elastic.

which are proportional to the rotations of an element $dx\,dy\,dz$ from its position of equilibrium about the three axes respectively. This amounts to ascribing to the particles of the medium a species of *polarity*, tending to place three orthogonal axes in each particle parallel respectively to the three corresponding axes in each of the other particles: the rotative force acting between the corresponding axes in each pair of particles being a function of the projection of the relative angular displacement of the axes on the plane passing through them, of the position of that plane, and of the distance between the particles.

A portion of a medium endowed with such molecular forces only, would transmit oscillations; but it would not tend to preserve any definite bulk or figure, nor would it resist any change of bulk or figure. It would be a *medium* or *system*, but not a *body*. Molecular forces of this kind, therefore, are not comprehended under the term *elasticity;* and the limits of the present investigation exclude those forms of the function V which represent the laws of their action.

29. The inquiry being thus restricted to molecular forces dependent on the variations of the bulk and figure of bodies, there is to be introduced a

Third Postulate: That supposing the body to be divided mentally into small parts, which, in the undisturbed state of the body are rectangular and of equal size, those parts, in the disturbed state, continue to be *sensibly* of equal bulk and similar figure, throughout a distance round each point at least equal to the greatest extent of appreciable molecular action.

This assumption has been made in all previous investigations, except those respecting the dispersion of light; and it seems, indeed, to be perfectly consistent with the real state of tangible bodies.

Its advantage in calculation is, that it enables us to treat the variations of the molecular forces acting on a given particle, as functions simply of the variations of bulk and figure of an originally rectangular element situated at that particle: seeing that the adjoining elements throughout the extent of appreciable molecular action continue always to undergo sensibly the same variations of bulk and figure as the element under consideration.

Let x_0, y_0, z_0 be the coordinates of any physical point in a homogeneous body in equilibrio, and whose particles

are not operated upon by any extraneous forces X, Y, Z. In this condition it is evident that

$$\delta V = 0$$

at every point, and that we may also make

$$V = 0.$$

In the disturbed condition, let ξ, η, ζ be the displacements of the point whose undisturbed position is (x_0, y_0, z_0), so that

$$x = x_0 + \xi, \text{ \&c.}$$

Then all the variations of bulk and figure which can be undergone by an originally rectangular element, consistently with the third postulate, may be expressed by means of the following six quantities, which I have elsewhere called *strains:*

$$\frac{d\xi}{dx} = \alpha; \quad \frac{d\eta}{dy} = \beta; \quad \frac{d\zeta}{dz} = \gamma;$$

$$\frac{d\eta}{dz} + \frac{d\zeta}{dy} = \lambda; \quad \frac{d\zeta}{dx} + \frac{d\xi}{dz} = \mu; \quad \frac{d\xi}{dy} + \frac{d\eta}{dx} = \nu;^*$$

of which α, β, γ, are longitudinal *extensions* if positive, *compressions* if negative, and λ, μ, ν, are *distortions* in the planes perpendicular to x, y, z, respectively.

Hence it appears that

$$V = \phi(\alpha, \beta, \gamma, \lambda, \mu, \nu) \ldots\ldots\ldots\ldots (18).$$

30. The first assumption, that we may treat the body in calculation as composed of rectangular elements $\rho\, dx\, dy\, dz$, involves the consequence, that we may express all the molecular forces which act on each such element, by means of pressures, normal and tangential, exerted on its six faces. Taking yz, zx, xy, to denote the position of the faces of such an element, P to denote generally a normal pressure expressed in units of force per unit of area, and Q a tangential pressure similarly expressed, let the nine component pressures on unity of area of those faces be thus denoted:

Position of Face.	*Direction of Pressure.*		
	x	y	z
yz	P_1	Q_3	Q_2'
zx	Q_3'	P_2	Q_1
xy	Q_2	Q_1'	P_3

* This notation is substituted for

$$N_1, N_2, N_3, 2T_1, 2T_2, 2T_3,$$

as being more convenient.

By the definition of elasticity all pressures are excluded except those whose variations produce and are produced by variations of volume and figure of the parts of the body. Hence the pressures

$$Q_1 - Q_1'; \quad Q_2 - Q_2'; \quad Q_3 - Q_3';$$

whose tendency is to make the element $dx\,dy\,dz$ rotate about its three axes respectively, without change of form, must be null; and therefore

$$Q_1 = Q_1; \quad Q_2 = Q_2'; \quad Q_3 = Q_3' \ldots\ldots\ldots (19).$$

Mr. Haughton correctly remarks that this often quoted theorem of Cauchy is not true for all conceivable media. It is not true, for instance, for a medium such as that which Mr. MacCullagh assumed to be the means of transmitting light. It is true, nevertheless, for all molecular pressures which properly fall under the definition of elasticity, if that term be confined to the forces which preserve the figure and volume of bodies.

Let us now express the sum of the virtual moments of the molecular forces acting on the element $dx\,dy\,dz$, in terms of the pressures P_1, &c.; to do which, we must multiply each pressure by the virtual variation of the effect which it tends to produce in its own direction. Thus we obtain the following result:

$$\delta V = P_1\delta a + P_2\delta\beta + P_3\delta\gamma + Q_1\delta\lambda + Q_2\delta\mu + Q_3\delta\nu \ldots (20).$$

Hence the function V bears the following relations to the normal and tangential pressures at the faces of a rectangular element:

$$\left.\begin{aligned} P_1 &= \frac{dV}{da}; \quad P_2 = \frac{dV}{d\beta}; \quad P_3 = \frac{dV}{d\gamma} \\ Q_1 &= \frac{dV}{d\lambda}; \quad Q_2 = \frac{dV}{d\mu}; \quad Q_3 = \frac{dV}{d\nu} \end{aligned}\right\} \ldots\ldots (21).$$

31. A *Fourth Postulate*, generally assumed in investigations of this kind, is that the pressures are sensibly proportional simply to the strains with which they are connected. This assumption must be approximately true of any law of molecular action, when the pressures and strains are sufficiently small. It is known to be sensibly true for almost all bodies, so long as the pressures and strains are not so great as to impair their power of recovering their original volume and figure.

According to this postulate, the pressures P_1, &c. are algebraic functions of the first order of the strains a, &c.;

and consequently V is an algebraic function of the second order of those strains. The constant part of V, as we have already seen (Art. 29), is null.

Following the notation adopted by Mr. Haughton, let (a) denote the coefficient of a in V, (a^2) that of $-\frac{a^2}{2}$, $(\beta\gamma)$ that of $-\beta\gamma$, &c. Then

$$\begin{aligned} V = {} & (a)\,a + (\beta)\,\beta + (\gamma)\,\gamma + (\lambda)\,\lambda + (\mu)\,\mu + (\nu)\,\nu \\ & - (a^2)\frac{a^2}{2} - (\beta^2)\frac{\beta^2}{2} - (\gamma^2)\frac{\gamma^2}{2} - (\lambda^2)\frac{\lambda^2}{2} - (\mu^2)\frac{\mu^2}{2} - (\nu^2)\frac{\nu^2}{2} \\ & - (\beta\gamma)\,\beta\gamma - (\gamma a)\,\gamma a - (a\beta)\,a\beta - (\mu\nu)\,\mu\nu - (\nu\lambda)\,\nu\lambda - (\lambda\mu)\,\lambda\mu \\ & \quad - (a\lambda)\,a\lambda - (a\mu)\,a\mu - (a\nu)\,a\nu \\ & \quad - (\beta\lambda)\,\beta\lambda - (\beta\mu)\,\beta\mu - (\beta\nu)\,\beta\nu \\ & \quad - (\gamma\lambda)\,\gamma\lambda - (\gamma\mu)\,\gamma\mu - (\gamma\nu)\,\gamma\nu \ldots\ldots\ldots\ldots (22). \end{aligned}$$

The six coefficients of the terms of the first order in this equation, obviously represent the pressures, uniform throughout the whole extent of the body, to which it is subjected when its particles are in those positions from which the displacements are reckoned: that is to say, when

$$\xi = 0\,; \quad \eta = 0\,; \quad \zeta = 0.$$

Let $P_{1,0}$ &c. denote those pressures. Then

$$\left.\begin{aligned} (a) = P_{1,0}\,; \quad (\beta) = P_{2,0}\,; \quad (\gamma) = P_{3,0} \\ (\lambda) = Q_{1,0}\,; \quad (\mu) = Q_{2,0}\,; \quad (\nu) = Q_{3,0} \end{aligned}\right\} \ldots\ldots (23).$$

The *twenty-one* coefficients of the terms of the second order are the *coefficients of elasticity* of the body, as referred to the three axes selected. The negative sign is prefixed to each, because it is essential to the stability of a body, that molecular pressures should be opposite in direction to the strains producing them.

The transformation of the quantities in equation (22) for any set of rectangular axes, is effected by means of equation (2) of § I. Art. 9, by making the following substitutions:

for	P_1,	P_2,	P_3,	$2Q_1$,	$2Q_2$,	$2Q_3$,
substitute	a,	β,	γ,	λ,	μ,	ν,

and make similar substitutions for the accented symbols. By multiplying the six equations referred to together by pairs, twenty-one equations are obtained, serving to transform the squares and products of α, β, γ, λ, μ, ν. Formulæ similar

to those which transform the strains a, &c. and their half-squares and products, serve also to transform the respective coefficients of those quantities in equation (22).*

It is shewn by Mr. Haughton, that by properly selecting the axes of co-ordinates, the number of independent coefficients of elasticity may always be reduced to three less than when the axes are indefinite; and by Mr. Haughton and Mr. Green, that when the body has orthogonal axes of elasticity at each point, then, if those axes be taken as the axes of co-ordinates, the coefficients of elasticity are reduced to the first nine. The latter proposition is obvious, because if molecular action be symmetrical about three orthogonal planes, and those be taken for coordinate planes, then the value of that part of the function V which is of the second order cannot be altered by a change in the *sign* of either of the distortions λ, μ, ν; so that the coefficients of the last twelve terms of equation (22) must each be null.

The nine coefficients of elasticity of a body in those circumstances have the following values, in terms of the notation of the previous sections:

$$\left.\begin{array}{c} \textit{Coefficients of Longitudinal Elasticity.} \\ (a^2) = A_1; \quad (\beta^2) = A_2; \quad (\gamma^2) = A_3. \\ \textit{Coefficients of Lateral Elasticity.} \\ (\beta\gamma) = B_1; \quad (\gamma a) = B_2; \quad (a\beta) = B_3. \\ \textit{Coefficients of Rigidity.} \\ (\lambda^2) = C_1; \quad (\mu^2) = C_2; \quad (\nu^2) = C_3. \end{array}\right\} \ldots\ldots(24).$$

32. Def. *Let the term* Perfect Fluid *be used to denote the state of a body, which under a given uniform normal pressure, and at a given temperature, tends to preserve, and if disturbed to recover, a certain bulk; but offers no resistance to change of figure.*

In such a body, if the element whose original bulk was $dx\,dy\,dz$, becomes of the bulk $(1 + \sigma)\,dx\,dy\,dz$ (σ being a small fraction), we shall have

$$\sigma = a + \beta + \gamma,$$

and the function V must be of the form

$$V = P_0\sigma - (\sigma^2)\frac{\sigma^2}{2} \ldots\ldots\ldots\ldots(25),$$

where P_0 is the uniform normal pressure when the particles are not displaced, and (σ^2) a coefficient of elasticity, whose

* See the note at the end of this paper.

value, in the notation of the previous sections, is :

Coefficient of Fluid Elasticity.

$$(\sigma^2) = (\alpha^2) = (\beta^2) = (\gamma^2) = (\beta\gamma) = (\gamma\alpha) = (\alpha\beta) = J \ldots (26).$$

The normal pressure in the disturbed state, which is the same in all directions, is obviously

$$P_1 = P_2 = P_3 = P_0 - J\sigma \ \ldots\ldots\ (27).$$

The tangential pressures are each = 0.

SECT. VII.—*On the proof of the Laws of Elasticity by the Method of Virtual Velocities.*

33. Having thus followed very nearly the steps of the researches of Mr. Haughton and Mr. Green, so as to compare their coefficients with those used in the previous part of this paper, I shall now investigate how far the method of Lagrange can be used to establish those relations between the coefficients of elasticity of different kinds in homogeneous solid bodies, which I have elsewhere deduced from geometrical and physical considerations.

The fluid elasticity considered in the last article cannot arise from the mutual actions of centres of force; for such actions would necessarily tend to preserve a certain arrangement amongst those centres, and would therefore resist change of figure. Fluid elasticity must arise either from the mutual actions of the parts of continuous matter or from the centrifugal force of molecular motions, or from both those causes combined.

On the other hand, it is only by the mutual action of centres of force that resistance to change of figure and molecular arrangement can be explained, that property being inconceivable of a continuous body. The elasticity *peculiar* to solid bodies is therefore due to the mutual action on centres of force. Solid bodies may nevertheless possess, in addition, a portion of that species of elasticity which belongs to fluids.

The investigation is simplified by considering in the first place the elasticity of a solid body as arising from the mutual action of centres of force only, and afterwards adding the proper portion of fluid elasticity.

It is known that solid bodies are capable of preserving bulk and figure, although their surfaces are acted upon by no sensible pressure, normal or tangential. We may take the positions of the particles in this condition, as points from which to measure their displacements. Thus we cause the

coefficients of all the terms of the first order in equation (22) to vanish.

To investigate the properties of the coefficients of elasticity, the function δV is to be expressed in a new form, viz. as the sum of the virtual moments of the actions exerted upon each of the centres of force in the particle under consideration, by the centres of force in all the other particles. Mr. Haughton, in his first memoir, having performed this process, shews by means of its results, that in a body composed entirely of centres of force acting along the lines joining them, the number of independent coefficients of elasticity for any system of orthogonal axes is reduced to *fifteen*, which by properly selecting those axes, may be reduced, for bodies in general, to *twelve*, and for those having axes of elasticity, to *six*.

I shall now endeavour to prove by the method of virtual velocities, what I have in the third section proved by other modes of reasoning; that in a homogeneous body constituted of centres of force only, the independent coefficients of elasticity are reducible to *three*, of which, and of the position of the axes, the twenty-one in equation (22) are functions.

A fourth independent coefficient is to be added, in solids possessing a portion of fluid elasticity; that is to say, in all known solids.

34. It is known that a homogeneous solid can exist, with its particles in an unstrained condition, bounded by plane surfaces in any direction. In this condition, therefore, the total molecular action upon a particle situated at any bounding plane must be null. Conceive the bounding plane still to pass through the same particle, but to have its position shifted through any angle. The molecular action on the particle will still be null. Now the effect of the shifting of the bounding plane is to take away a wedge of matter from one side of the particle, and to substitute an equal and similarly constituted wedge, lying in a diametrically opposite direction. Hence, in the unstrained condition of a solid body, the action exercised upon any particle, by a wedge of matter bounded by any two planes passing through the particle, is null.

This shews that the action of a wedge of solid matter on a particle situated at its edge is not altered by varying the angular position of the wedge; and consequently, that the molecular actions which produce elasticity are not *directly* functions of the relative angular positions of the centres of

force which act on each other, but merely of their distances apart, so that if the actions of the several equal wedges into which a body may be conceived to be divided, round a given particle, are different, this does not arise directly from the angular positions of the wedges, but from the different distribution of their centres of force as to *distance* from those of the particle operated upon.

(I have proved this, in a manner slightly different in form, in a supplementary paper to § III., Art. 17).

This further shews, that the mutual action of two centres of force in a solid must be directed along the line joining them; for otherwise it would tend to bring that line into some definite angular position, and would be a function of the direction of the line.

It finally results, from what has been stated, that the action of an indefinitely slender pyramid of a solid body upon a particle at its apex must be a direct attraction or repulsion along the axis of the pyramid, which is a function of the several distances of the centres of force in the pyramid from those in the particle at the apex, and which, in the unstrained condition of the body, must be null.

The principles stated above have to a greater or less extent been taken for granted in previous investigations, but have not hitherto been demonstrated. They may all be regarded as the necessary consequences of the following:

DEF. *Let the term* ELASTIC SOLID *be used to denote the condition of a body, which, when acted upon by any given system of pressures, or by none, and at a given temperature, tends to preserve, and if disturbed, to recover, a definite bulk and figure; and such that, if while in an unstrained condition it be cut into parts of any figure, those parts, when separate, will tend to preserve the same bulk and figure as they did when they formed one body.*

Experience informs us that bodies *sensibly* agreeing with this definition exist; its consequences are therefore applicable to them in practice.

35. Let r denote the distance apart of two centres of force in an unstrained solid, and let ϕr be proportional to their mutual action. Then

$$dx\,dy\,dz\,d^2\omega.\Sigma\phi r = 0$$

may be taken to represent the total action of an indefinitely slender pyramid which subtends the element of angular space $d^2\omega$ upon a particle at its apex $dx\,dy\,dz$.

In consequence of a strain, let each of the distances r become

$$(1+\varepsilon)r,$$

ε being a very small fraction. Then the total action of the pyramid becomes

$$dxdydzd^2\omega.\Sigma(\phi r + \varepsilon r\phi' r) = dxdydzd^2\omega.\varepsilon\Sigma r\phi' r\,;$$

for by the third postulate, ε is uniform throughout the extent of appreciable molecular action.

The quantity which the force acting between two centres of force tends to vary, is their relative displacement along the line joining them, or εr. Hence the sum of the virtual moments of the actions of all the slender pyramids into which the solid is conceived to be divided, that is to say, the total virtual moment of its molecular action, upon the particle $dxdydz$, is

$$\delta V dxdydz = dxdydz \iint \varepsilon\delta\varepsilon.\Sigma(r^2\phi' r).d^2\omega,$$

the double integration extending to all angular space. Consequently, we obtain as a new value of the function V,

$$V = \iint \tfrac{1}{2}\varepsilon^2\ \Sigma(r^2\phi' r).d^2\omega \ldots\ldots\ldots (28).$$

Let a, b, c be the direction-cosines of the axis of a given slender pyramid. Then it is easily seen that the strain ε along that axis has the following value in terms of the six strains as referred to the axes of coordinates

$$\varepsilon = \alpha a^2 + \beta b^2 + \gamma c^2 + \lambda bc + \mu ca + \nu ab,$$

and consequently that

$$\begin{aligned}\frac{\varepsilon^2}{2} = {} & \frac{\alpha^2}{2}a^4 + \frac{\beta^2}{2}b^4 + \frac{\gamma^2}{2}c^4 + \frac{\lambda^2}{2}b^2c^2 + \frac{\mu^2}{2}c^2a^2 + \frac{\nu^2}{2}a^2b^2 \\ & + \beta\gamma b^2c^2 + \gamma\alpha c^2a^2 + \alpha\beta a^2b^2 \\ & + \mu\nu a^2bc + \nu\lambda ab^2c + \lambda\mu abc^2 \\ & + \alpha\lambda a^2bc + \alpha\mu a^2c + \alpha\nu a^2b \\ & + \beta\lambda b^3c + \beta\mu ab^2c + \beta\nu ab^3 \\ & + \gamma\lambda bc^3 + \gamma\mu ac^3 + \gamma\nu abc^2\end{aligned}$$

If this value of $\frac{1}{2}\varepsilon^2$ be substituted in equation (28), and the result compared with equation (22), it is at once obvious that the twenty-one coefficients of elasticity have the following values (putting $\Sigma(r^2\phi' r) = -R$, which is negative that equilibrium may be stable):

$$\left.\begin{aligned}
(\alpha^2) &= \iint a^4 R d^2\omega \\
(\beta^2) &= \iint b^4 R d^2\omega \\
(\gamma^2) &= \iint c^4 R d^2\omega \\
(\lambda^2) &= (\beta\gamma) = \iint b^2 c^2 R d^2\omega \\
(\mu^2) &= (\gamma\alpha) = \iint c^2 a^2 R d^2\omega \\
(\nu^2) &= (\alpha\beta) = \iint a^2 b^2 R d^2\omega \\
(\mu\nu) &= (\alpha\lambda) = \iint a^2 bc R d^2\omega \\
(\nu\lambda) &= (\beta\mu) = \iint ab^2 c R d^2\omega \\
(\lambda\mu) &= (\gamma\nu) = \iint abc^2 R d^2\omega \\
(\beta\lambda) &= \iint b^3 c R d^2\omega \\
(\gamma\lambda) &= \iint bc^3 R d^2\omega \\
(\gamma\mu) &= \iint c^3 a R d^2\omega \\
(\alpha\mu) &= \iint ca^3 R d^2\omega \\
(\alpha\nu) &= \iint a^3 b R d^2\omega \\
(\beta\nu) &= \iint ab^3 R d^2\omega
\end{aligned}\right\} \quad \ldots\ldots\ldots (29).$$

In the above equations, which agree with those given by Mr. Haughton, the number of independent coefficients is *fifteen*.

36. Their reduction to a smaller number arises from the nature of the function

$$R = -\Sigma(r^2\phi' r).$$

This quantity is a function of the distances of the centres of force in a given indefinitely slender pyramid from those in a particle at its apex, and can vary with the direction-cosines a, b, c of the axis of the pyramid, solely because those distances vary with them. Now in a homogeneous solid, that is, one composed of a succession of similar and regularly placed groups of centres of force, those distances depend upon a quantity which may be called the *mean interval* between the centres of force in a given direction: a quantity of such a nature that the product of its three values for any three orthogonal directions is a constant quantity; being the space occupied by a centre of force, or by a definite group of such centres. To have this property, the mean interval must be a quantity of this form:

$$i = e^{f+pa^2+qb^2+kc^2+lbc+mca+nab} \quad \ldots\ldots\ldots (30);$$

that is to say, its logarithm must be proportional to the

reciprocal of the square of the radius of an ellipsoid, whose axes are those of molecular arrangement, and therefore of molecular action, and of elasticity.

Let the axes of this ellipsoid be taken as axes of coordinates. Then $l = 0$, $m = 0$, $n = 0$; and the above equation is reduced to

$$i = e^{f+ga^2+hb^2+kc^2} \quad \ldots\ldots\ldots\ldots (30A),$$

and because the quantity

$$R = F(i) = \psi(f + ga^2 + hb^2 + kc^2) \quad \ldots\ldots (31)$$

does not change its sign or value by any change of the signs of the cosines a, b, c, it follows that all the coefficients in (29) containing odd powers of those cosines, that is to say, all except the *first six*, disappear when the axes of molecular arrangement are taken for axes of coordinates.

These six, for all known homogeneous substances, are reducible to *three*, by the following reasoning:

Let us assume as a *Fifth Postulate*, what experience shews to be sensibly true of all known homogeneous substances, viz. that their elasticity varies very little in different directions. Those substances, such as timber, whose elasticity in different directions varies much, are not homogeneous, but composed of fibres, layers, and tubes of different substances.

If this be assumed, it follows that, in the expression (31), for the quantity R, the variable terms

$$ga^2 + hb^2 + kc^2$$

are very small compared with the constant term f, and that R may be developed in the form

$$R = \psi(f) + \psi'(f).(ga^2 + hb^2 + kc^2) + \&c.$$

If this value of R be introduced into equation (29), and if small quantities of the second order be neglected, it is easily seen, on performing the integrations, that the following relations exist amongst the six coefficients already specified:

$$\left.\begin{aligned} (\beta^2) + (\gamma^2) &= 6(\lambda^2) = 6(\beta\gamma) \\ (\gamma^2) + (\alpha^2) &= 6(\mu^2) = 6(\gamma\alpha) \\ (\alpha^2) + (\beta^2) &= 6(\nu^2) = 6(\alpha\beta) \end{aligned}\right\}$$

or, by transformation,

$$\left.\begin{aligned} (\alpha^2) &= 3\{(\mu^2) + (\nu^2) - (\lambda^2)\} \\ (\beta^2) &= 3\{(\nu^2) + (\lambda^2) - (\mu^2)\} \\ (\gamma^2) &= 3\{(\lambda^2) + (\mu^2) - (\nu^2)\} \end{aligned}\right\} \quad \ldots\ldots\ldots (32).$$

These equations reduce the number of independent coefficients of elasticity arising from the actions of centres of force, to *three*. They are identical with the equations (7) and (8), embodied in the fifth theorem in § III., although arrived at by a different process.

37. Let us suppose the solid under consideration to possess a portion of fluid elasticity, represented by the coefficient J. Then the coefficients of elasticity have evidently the following relations:

$$\left.\begin{aligned}
(\alpha^2) &= 3\{(\mu^2) + (\nu^2) - (\lambda^2)\} + J \\
(\beta^2) &= 3\{(\nu^2 + (\lambda^2) - (\mu^2)\} + J \\
(\gamma^2) &= 3\{(\lambda^2) + (\mu^2) - (\nu^2)\} + J \\
(\beta\gamma) &= (\lambda^2) + J \\
(\gamma\alpha) &= (\mu^2) + J \\
(\alpha\beta) &= (\nu^2) + J
\end{aligned}\right\} \quad \ldots\ldots (33),$$

which are identical with the six equations (9) comprehended under the sixth theorem, in § IV.

38. The Laws of Elasticity stated in this paper, are the necessary consequences of the definitions of elasticity and of fluid and solid bodies, given in Arts. 28, 32, and 34, respectively, when taken in conjunction with five postulates or assumptions, which however may be summed up in two, viz.

First, That the variations of molecular force concerned in producing elasticity are sufficiently small to be represented by functions of the first order of the quantities on which they depend: and

Secondly, That the integral calculus and the calculus of variations are applicable to the theory of molecular action. It is thus apparent that the science of elasticity is to a great extent one of deduction *à priori*.

The functions of perceptive experience in connexion with it are twofold: first, by observation, to inform us of the existence of substances, agreeing to a greater or less degree of approximation with the definitions and postulates; and secondly, by experiment, to ascertain the numerical value of the coefficients of elasticity of each substance.

59, St. Vincent Street, Glasgow,
March 8, 1852.

NOTE TO SECTIONS VI. AND VII. OF PRECEDING PAPER.

By W. J. M. Rankine.

On the Transformation of Coefficients of Elasticity, by the aid of a Surface of the Fourth Order.

(The following note contains no original principle, and is designed merely to put upon record, for the sake of convenient reference, a series of equations which will be found useful in future investigations.)

It has been pointed out by Mr. Haughton, in his first paper, that if we take into consideration that part only of the elasticity of a solid which arises from the mutual actions of centres of force, so that the function V shall contain at most but fifteen unequal coefficients (viz. those whose values are given in equation 29), and if, with those fifteen coefficients, we construct a surface of the fourth order, whose equation is the following,

$$\begin{aligned} U = {} & (a^2)\, x^4 + (\beta^2)\, y^4 + (\gamma^2)\, z^4 \\ & + 6\,(\lambda^2)\, y^2z^2 + 6\,(\mu^2)\, z^2x^2 + 6\,(\nu^2)\, x^2y^2 \\ & + 12\,(a\lambda)\, x^2yz + 12\,(\beta\mu)\, xy^2z + 12\,(\gamma\nu)\, xyz^2 \\ & + 4\,(a\nu)\, x^3y + 4\,(a\mu)\, x^3z \\ & + 4\,(\beta\lambda)\, y^3z + 4\,(\beta\nu)\, y^3x \\ & + 4\,(\gamma\mu)\, z^3x + 4\,(\gamma\lambda)\, z^3y \\ = {} & 1 \qquad \ldots\ldots\ldots\

Let the relative direction-cosines of the two sets of rectangular axes be expressed as follows:

Original Axes.	*New Axes.* x'	y'	z'	
x	a_1	b_1	c_1	} cosines.
y	a_2	b_2	c_2	
z	a_3	b_3	c_3	

Let the following notation be used for functions of those cosines. (It is the same which is employed by Mr. Haughton):

$$p_1 = b_2c_3 + b_3c_2; \quad p_2 = b_3c_1 + b_1c_3; \quad p_3 = b_1c_2 + b_2c_1;$$
$$q_1 = c_2a_3 + c_3a_2; \quad q_2 = c_3a_1 + c_1a_3; \quad q_3 = c_1a_2 + c_2a_1;$$
$$r_1 = a_2b_3 + a_3b_2; \quad r_2 = a_3b_1 + a_1b_3; \quad r_3 = a_1b_2 + a_2b_1;$$

$$l_1 = b_2c_3 + b_3c_2; \quad l_2 = b_3c_1 + b_1c_3; \quad l_3 = b_1c_2 + b_2c_1;$$
$$m_1 = c_2a_3 + c_3a_2; \quad m_2 = c_3a_1 + c_1a_3; \quad m_3 = c_1a_2 + c_2a_1;$$
$$n_1 = a_2b_3 + a_3b_2; \quad n_2 = a_3b_1 + a_1b_3; \quad n_3 = a_1b_2 + a_2b_1;$$

then the following are the six equations of transformation of the second order for the surface $U = 1$,

$$x'^2 = x^2a_1^2 + y^2a_2^2 + z^2a_3^2 + 2yza_2a_3 + 2zxa_3a_1 + 2xya_1a_2,$$

(for y'^2, z'^2, similar equations in b, c, respectively).

$$y'z' = x^2p_1 + y^2p_2 + z^2p_3 + yzl_1 + zxl_2 + xyl_3,$$

(for $z'x'$, a similar equation in q and m),

(for $x'y'$, a similar equation in r and n) ..... (C).

Those equations are made applicable to the transformation of strains by the following substitutions:

for x^2, y^2, z^2, $2yz$, $2zx$, $2xy$,

substitute a, β, γ, λ, μ, ν;

and to that of pressures, by the following:

for x^2, y^2, z^2, yz, zx, xy,

substitute P_1, P_2, P_3, Q_1, Q_2, Q_3;

and similar substitutions for the accented symbols.

The following are the fifteen equations of transformation of the fourth order:

$$x'^4 = x^4a_1^4 + y^4a_2^4 + z^4a_3^4 + 6y^2z^2a_2^2a_3^2 + 6z^2x^2a_3^2a_1^2 + 6x^2y^2a_1^2a_2^2$$
$$+ 12x^2yza_1^2a_2a_3 + 12xy^2za_1a_2^2a_3 + 12xyz^2a_1a_2a_3^2$$
$$+ 4x^3ya_1^3a_2 + 4x^3za_1^3a_3 + 4y^3za_2^3a_3 + 4y^3xa_2^3a_1 + 4z^3xa_3^3a_1 + 4z^3ya_3^3a_2;$$

(for y'^4, z'^4, similar equations in b, c, respectively).

$$y'^2z'^2 = x^4p_1^2 + y^4p_2^2 + z^4p_3^2 + y^2z^2(l_1^2 + 2p_2p_3) + z^2x^2(l_2^2 + 2p_3p_1) + x^2y^2(l_3^2 + 2p_1p_2)$$
$$+ 2x^2yz(p_1l_1 + l_2l_3) + 2xy^2z(p_2l_2 + l_3l_1) + 2xyz^2(p_3l_3 + l_1l_2)$$
$$+ 2x^3yp_1l_3 + 2x^3zp_1l_2 + 2y^3zp_2l_1 + 2y^3xp_2l_3 + 2z^3xp_3l_2 + 2z^3yp_3l_1;$$

(for $z'^2x'^2$, a similar equation in q and m);

(for $x'^2y'^2$, a similar equation in r and n);

$$
\begin{aligned}
x^2y'z' &= x^4a_1^2p_1 + y^4a_3^2p_2 + z^4a_2^2p_3 \\
&\quad + y^2z^2(a_2^2p_3 + a_3^2p_2 + 2a_2a_3l_1) \\
&\quad + z^2x^2(a_3^2p_1 + a_1^2p_3 + 2a_3a_1l_2) \\
&\quad + x^2y^2(a_1^2p_2 + a_2^2p_1 + 2a_1a_2l_3) \\
&\quad + x^2yz(a_1^2l_1 + 2a_2a_3p_1 + 2a_2a_1l_3 + 2a_1a_3l_2) \\
&\quad + xy^2z(a_2^2l_2 + 2a_3a_1p_2 + 2a_1a_2l_1 + 2a_2a_3l_3) \\
&\quad + xyz^2(a_3^2l_3 + 2a_1a_2p_3 + 2a_2a_3l_2 + 2a_3a_1l_1) \\
&\quad + x^2y(a_1^2l_3 + 2a_1a_2p_1) + x^2z(a_1^2l_2 + 2a_1a_3p_1) \\
&\quad + y^2z(a_2^2l_1 + 2a_2a_3p_2) + y^2x(a_2^2l_3 + 2a_2a_1a_2) \\
&\quad + z^2x(a_3^2l_2 + 2a_3a_1p_3) + z^2y(a_3^2l_1 + 2a_3a_2p_3);
\end{aligned}
$$

(for $x'y^2z'$, a similar equation in b, q, m);

(for $x'y'z^2$, a similar equation in c, r, n);

$$
\begin{aligned}
x^2y' &= x^4a_1^2r_1 + y^4a_2^2r_2 + z^4a_3^2r_3 \\
&\quad + 3y^2z^2a_2a_3n_1 \\
&\quad + 3z^2x^2a_3a_1n_2 \\
&\quad + 3x^2y^2a_1a_2n_3 \\
&\quad + 3x^2yz(a_1^2n_1 + 2a_2a_3r_1) \\
&\quad + 3xy^2z(a_2^2n_2 + 2a_3a_1r_2) \\
&\quad + 3xyz^2(a_3^2n_3 + 2a_1a_2r_3) \\
&\quad + x^2y(a_1^2n_3 + 2a_1a_2r_1) + x^2z(a_1^2n_2 + 2a_3a_1r_1) \\
&\quad + y^2z(a_2^2n_1 + 2a_2a_3r_2) + y^2x(a_2^2n_3 + 2a_1a_2r_2) \\
&\quad + z^2x(a_3^2n_2 + 2a_3a_1r_3) + z^2y(a_3^2n_1 + 2a_2a_3r_3);
\end{aligned}
$$

(for x^2z', a similar equation in a, q, and m);

(for y^2z', a similar equation in b, p, and l);

(for y^2x', a similar equation in b, r, and n);

(for z^2x', a similar equation in c, q, and m);

(for z^2y', a similar equation in c, p, and l) .....(*D*).

The above equations are made applicable to the transformation of the coefficients of elasticity arising from the mutual actions of centres of force only, by the following substitutions:

for x^4, y^4, z^4, y^2z^2, z^2x^2, x^2y^2,
substitute (a^2), (β^2), (γ^2), $(\beta\gamma) = (\lambda^2)$, $(\gamma a) = (\mu^2)$, $(a\beta) = (\nu^2)$;

for x^2yz, xy^2z, xyz^2,
substitute $(a\lambda) = (\mu\nu)$, $(\beta\mu) = (\nu\lambda)$, $(\gamma\nu) = (\lambda\mu)$;

for x^2y, x^2z, y^2z, y^2x, z^2x, z^2y,
substitute $(a\nu)$, $(a\mu)$, $(\beta\lambda)$, $(\beta\nu)$, $(\gamma\mu)$, $(\gamma\lambda)$;

and similar substitutions for the accented symbols.

Should the substance under consideration be endowed with a portion of fluid elasticity in addition to that which arises from the mutual action of centres of force, the coefficient of that fluid elasticity J must be subtracted from the coefficients into which it enters; viz.

(a^2), (β^2), (γ^2), $(\beta\gamma) = (\lambda^2) + J$, $(\gamma a) = (\mu^2) + J$, $(a\beta) = (\nu^2) + J$,

before effecting the transformation.

The results of the transformation for those six coefficients, being increased by the same quantity J which was previously subtracted, will give their entire values for the new axes.

If the original axes of coordinates are those of elasticity, each of the fifteen equations of transformation is reduced to its first six terms, in which the following substitutions are to be made for the unaccented symbols:

for x^4, y^4, z^4, y^2z^2, z^2x^2, x^2y^2,

substitute $A_1 - J$, $A_2 - J$, $A_3 - J$, $B_1 - J = C_1$, $B_2 - J = C_2$, $B_3 - J = C_3$.

59, *St. Vincent Street, Glasgow,*
April 7, 1852.

ON THE DOCTRINE OF IMPOSSIBLES IN ALGEBRAIC GEOMETRY.

By William Walton.

In a work of great value *On the Higher Plane Curves*, by Mr. Salmon, which has recently appeared, may be seen a general critique on the theory of Impossibles, in which he has pronounced an unfavourable opinion in regard to the special views propounded some years ago by the late Mr. Gregory and by myself on the geometrical interpretation of impossibles. As I cannot, after a careful perusal of his remarks, assent to the validity of the grounds of his objections, I feel that I have no alternative but to express in this *Journal*, where the theory was originally published, a few observations on the points at issue. I may perhaps be allowed to observe that, from the very high regard which I entertain for the genius of this writer, I do not enter upon this question without duly considering the weight of his authority on matters of geometrical science.

The papers in the *Cambridge Mathematical Journal* which appear to form the subject of Mr. Salmon's critique, are the following, the first of them being Mr. Gregory's:

"On the Existence of Branches of Curves in several Planes." *May*, 1839.

"On the General Theory of the Loci of Curvilinear Intersection." *February*, 1840.

"On the General Interpretation of Equations between two variables in Algebraic Geometry." *May*, 1840.

"On the General Theory of Multiple Points." *November*, 1840.

"On the Existence of Possible Asymptotes to Impossible Branches of Curves." *February*, 1841.

I will now endeavour to state, as concisely as possible, Mr. Salmon's objections to the theory advocated in these papers, together with some of his inferences and remarks. Lest I should not have given an adequate abstract of his opinions, the reader is requested, if convenient, to consult his own work. I will commence, however, with transcribing a portion of Mr. Salmon's statement of Mr. Gregory's views, in order to render the observations of Mr. Salmon intelligible.

"It is to a certain extent arbitrary what interpretation we give to our algebraical equations; but the greatest advantage is gained when we adopt the most general methods, and when every algebraical symbol has its appropriate geometrical representation. Thus the inventors of analytic geometry might, if they pleased, have left the sign $-$ uninterpreted, and, confining their attention to the positive values of the variables, have only considered those branches of a curve which lie in the upper right-hand angle between the axes. It was soon seen, however, how much generality might be gained by interpreting the line $-a$ as of equal length, but opposite direction, to the line $+a$; and no curve is now considered as completely traced unless the negative, as well as the positive, values of the variables be taken into account. This, however, is merely a matter of convention, and we might, if we pleased, have restricted ourselves to the positive values of the coordinates."..."This system of interpretation" (viz. that given by the theory of impossibles developed in the articles named) "is quite as legitimate an extension as that of the negative values of the variables, and is as necessary to the thorough understanding the course of a curve."

The following is a list of Mr. Salmon's objections and some of his remarks:

(1). "Our method of interpretation is not so wholly conventional as Mr. Gregory represents. It is necessary, in the first place, that our interpretations should be consistent; thus, had we commenced by rejecting the negative values of the variables, we should have discovered the incompleteness of our method, by finding that on transformation of coordinates our equations had no longer the same geometrical signification."

(2). "We do not obtain our first knowledge of geometrical figures from interpreting the equations of analytic geometry; but, on the contrary, our interpretations of analytical equa-

tions must be made to coincide with our previous geometrical knowledge."....." We know what a circle is before we know anything about the equation $x^2 + y^2 = a^2$, and any interpretation of this equation differing either by defect or excess from our previous geometrical conception, must be rejected. We discover that we should be wrong in leaving the sign – uninterpreted, because then the equation $x^2 + y^2 = a^2$ would only represent a fragment of a circle; and we may in the same manner discover that it is objectionable to give a real interpretation to the symbol $\sqrt{(-1)}$ in the equations of analytic geometry, because then $x^2 + y^2 = a^2$ represents not only a circle but an irrelevant curve besides."

(3). "In Mr. Gregory's first papers, when he came to an imaginary value, he usually contented himself with saying that the curve left the plane of reference, without much troubling himself to inquire where it went to."

(4). "This omission was supplied in a paper by Mr. Walton, which I cannot but regard as an able *reductio ad absurdum* of Mr. Gregory's theory. He finds that, according to Mr. Gregory's principles, properly generalized, the equation $x^2 + y^2 = a^2$ represents not only a circle, but also a curve of the fourth degree in space, whose ordinary equations would be written

$$x^2 + y^2 - x^2 \tan^2 2r\pi - (z - x \tan 2r\pi)^2 = a^2 \cos 4m\pi,$$

$$2x(x - y) \tan 2r\pi + 2yz = a^2 \sin 4m\pi,$$

where, however, r is arbitrary; and we are as much at liberty to choose which of an infinity of such curves we are to regard as the companion of the circle, as in M. Poncelet's method we were left to choose between an infinity of equilateral hyperbolæ." (Mr. Salmon has given an account of M. Poncelet's method). "Now if it can be shewn that our ordinary conceptions of a circle are defective, and that one or all of these curves possess the same properties, and are entitled to be regarded as a portion of the same curve, then Mr. Gregory's mode of interpretation must be hailed as a great discovery. But if these curves differ from a circle in form and properties, then it is an abuse of language to speak of them as branches of a circle, merely because they can be represented by the same equation."

(5). "To talk of plane curves having branches out of their plane appears to me calculated to confound all a student's ideas, to make him likely to lose his hold of the principle

that a curve of the n^{th} degree is always cut by a right line in n points, and to make him fancy that he has distinct conceptions, which he cannot possibly have, of imaginary points and lines."

(6). "The explanation of conjugate points on this theory does not shew why these points should be always double points, and at any rate the relation of such a point (considered as the limit of an oval) to the curve, seems sufficiently intelligible."

(7). "The same theory has been applied to the explanation of an impossible branch of a curve having a real asymptote; but this is nothing more than a conjugate point at infinity."

Having now stated Mr. Salmon's observations, I will proceed to reply to them seriatim.

(1). Mr. Gregory's remarks on the use of the sign − are I conceive based on the supposition that our object is to interpret the geometrical signification of equations in algebraic geometry in accordance with the laws of combination of algebraic symbols and with the geometrical meanings primarily assigned to symbols of quantity and affection. The ancient geometricians, unacquainted with algebraic analysis, had no alternative but to discuss the properties of curves by geometrical reasonings. On the invention of the Cartesian method of geometry the primary object of geometricians was to express in an algebraic form the known geometrical properties of curves. A species of inverse investigation was yet open to the inquiries of mathematicians, viz. to define curves by algebraic formulæ, and thence to infer their forms by interpretation subjected to any or no restriction. Mr. Gregory's observations are I think obviously based upon the latter hypothesis, and I do not see how any objection can be properly urged against his assertion, that we are entitled, if we choose, to restrict ourselves to the exclusive use of the sign +, it following of course thereby that our equations have a more limited geometrical signification than they would have were the sign − also accepted. If we first of all define a circle geometrically, and then, using the equation $x^2 + y^2 = a^2$, confine ourselves to the sign +, calling the locus of the equation a circle, we no doubt contradict ourselves by creating incompatible definitions. But if we define a curve as the "locus of the equation $x^2 + y^2 = a^2$, on the supposition that the sign − is rejected," we certainly are guilty of no inconsistency. It may

be remarked moreover that our formulæ of transformation

$$x = x' \cos\theta - y' \sin\theta, \quad y = x' \sin\theta + y' \cos\theta,$$

are inapplicable on the hypothesis of our rejecting the sign –. If however, without employing these formulæ of transformation, we manually draw two fresh lines of reference, and proceed as before, rejecting the sign –, we merely do the same thing over again in another place, the logical necessity of using the sign – not resulting from the conception of such change of axes.

(2). The observations which I have made on the first division of Mr. Salmon's objections, constitute in a great measure a sufficient reply to the second division. His remarks are I think due to his confounding together two inverse processes of reasoning. It is undoubtedly too obvious to need an assertion, that if we take $x^2 + y^2 = a^2$ as the mere algebraic expression of a circular locus, thereby confining within certain limits the values of x and y, we shall be contradicting ourselves, if we make it, on the same hypothesis in regard to x and y, to denote either a circle and likewise something else, or something less than a circle. If however we do not call it "the equation to a circle," but "the equation to a locus including, it may be, a circle," which we may represent by it without violating any law of either geometry or algebra, we are liable to no such imputation.

(3). I am not aware of any earlier paper by Mr. Gregory on the theory of impossibles in application to curves, than the one to which I have referred, published in May, 1839. In this paper he gives a rule in itself clear and unambiguous for constructing the course of the curve on its departure from the plane of reference: a clearer statement however of his really clear rule I have given at the end of the paper in the May Number of 1840.

(4). In the paper to which Mr. Salmon refers, in which I have presented Mr. Gregory's views in a more general form, I have inferred from the laws of algebraical combination and the first principles of geometrical interpretation that the equation $x^2 + y^2 = a^2$ properly represents, in its widest significancy, according to one of an infinite number of admissible constructions of impossible axes, any one of an infinite number of curves of which the locus is a surface of the fourth degree. That the first principles of algebra and geometrical interpretation have been misrepresented by me, or that my inferences from them are illogical, Mr. Salmon has not attempted to shew. He seems to presume that in

writing the equation $x^2 + y^2 = a^2$, one must necessarily regard it as the equation to a circle, which I have not done, and proceeds accordingly to object to my inferences because the ordinary notion of a circle cannot be said to be defective. I have not called the series of curvilinear loci, denoted by this equation, branches of the circle, although I have shewn that they as well as the circle are represented by the equation. It is possible that Mr. Gregory or myself may have somewhere used improper or erroneous phraseology in treating of such questions: however, the views to which I adhere I have now explained. Any fault of language on the part of the advocates of the geometrical theory of impossibles cannot affect the real value of the principles of this branch of geometry.

(5). To talk of plane curves having branches out of their plane would be a very strange mode of speaking, if it be thereby signified that a curve can both lie entirely in one plane and partially out of it, and, even if such a paradox were not implied, it cannot be called a good expression. However, there can be no impropriety in saying that a curve may lie partly within one plane and partly without it. I do not see, moreover, why a student, supposed to be intelligent, should be unable to grasp one truth because another is presented to his notice. That a student should be able to regard as real what is truly imaginary or impossible cannot be expected; but that he should be able to do so when what is called imaginary or impossible is only technically so designated, being neither the one nor the other in fact, is not more than might be anticipated. Whether such a branch of geometry is a fit subject for a student's attention, considering the vast range of mathematics, must be left to the consideration of those who are concerned in elementary tuition.

(6). The explanation of conjugate points on this theory does not shew why these points should be always double points, because they need not be double points. If the equation be algebraic and free from radicals, then, by the theory of equations, we know that such points will be double, quadruple, &c., because impossible roots enter equations by pairs. If $f(x, y) = 0$ be not algebraical, but transcendental, a conjugate point, not double, but single, triple, &c., may easily present itself. In regard to the theory of the vanishing oval, it may be observed that, supposing a conjugate point to originate in a vanishing oval, the doctrine of impossibles

would not be inconsistent with this hypothesis, but would add something to it. In fact, branches of the curve emanating from different points of the finite oval would intersect at the point of the oval's evanescence. If however we adopt the theory of the vanishing oval to the exclusion of all other theory, how are we to explain the fact that $\frac{dy}{dx}$ is sometimes possible and sometimes impossible at a conjugate point? For examples, at the conjugate point $x = a$, $y = a^3c$, of the curve

$$(y - cx^3)^2 = (x - b)^5 (x - a)^6,$$

where a is supposed to be less than b, we see that $\frac{dy}{dx} = 3a^2c$; while, at the conjugate point $x = a$, $y = b$, of the curve

$$(y - b)^2 = (x - a)^2 (x - c),$$

where a is less than c,

$$\frac{dy}{dx} = \pm (a - c)^{\frac{1}{2}},$$

impossible values. A geometrical explanation of these analytical phenomena flows naturally from the doctrine of impossibles.

(7). Supposing that what I have called an asymptote to an impossible branch of a curve resolves itself into nothing more than a conjugate point at infinity, my reply to the preceding division of Mr. Salmon's remarks will be available as far as a vindication of the fundamental principles of the theory of impossibles is concerned. I do not however concur with Mr. Salmon in considering it to be nothing more than a conjugate point at infinity. I admit that it may be regarded, if we please, as a conjugate point of a peculiar kind. It may in fact be considered as a conjugate point at an infinite distance in a straight line, dependent in position upon the parameters of the curve, which passes within a finite distance from the origin, and of which the ordinates at an infinite distance differ by zero from those of the curve. Such lines are, under ordinary circumstances, called rectilinear asymptotes. In saying that we are at liberty, if we please, to take this view of possible asymptotes to impossible branches, I am virtually concurring in the opinion expressed by Mr. Gregory in an analogous question, that the inventors of analytic geometry might, if they pleased,

have left the sign – uninterpreted, thereby forfeiting the advantages of a more enlarged interpretation.

I will avail myself of this opportunity of developing the principles of the transformation of coordinates applicable either to possible or impossible values of the coordinates. I shall adopt the relations given in my paper of May, 1840, which connect the quantitive and affectional coordinates. Like principles would be applicable for the transformation of coordinates on Mr. Gregory's or any other legitimate method of constructing the impossible axes.

Let x, y, be the affectional coordinates of any point referred to any rectangular axes; x_1, y_1, those of the same point referred to any other such axes in the plane of reference. Let

$$x = (+)^r\alpha, \quad y = (+)^s\beta,$$
$$x_1 = (+)^{r_1}\alpha_1, \quad y = (+)^{s_1}\beta_1.$$

Let x', y', z', be the quantitive coordinates of the same point in relation to the former, and x_1', y_1', z_1', in relation to the latter axes. Then, θ being the angle between the axes of x' and x_1', we have

$$x' = x_1' \cos\theta - y_1' \sin\theta, \quad y' = x_1' \sin\theta + y_1' \cos\theta, \quad z' = z_1';$$

and therefore

$$\alpha \cos 2r\pi = \alpha_1 \cos 2r_1\pi \cos\theta - \beta_1 \cos 2s_1\pi \sin\theta \ \ldots\ (1),$$
$$\beta \cos 2s\pi = \alpha_1 \cos 2r_1\pi \sin\theta + \beta_1 \cos 2s_1\pi \cos\theta \ \ldots\ (2),$$
$$\alpha \sin 2r\pi + \beta \sin 2s\pi = \alpha_1 \sin 2r_1\pi + \beta_1 \sin 2s_1\pi.$$

From these three equations we see that

$$\tan 2r\pi(\alpha_1 \cos 2r_1\pi \cos\theta - \beta_1 \cos 2s_1\pi \sin\theta)$$
$$+ \tan 2s\pi(\alpha_1 \cos 2r_1\pi \sin\theta + \beta_1 \cos 2s_1\pi \cos\theta)$$
$$= \alpha_1 \sin 2r_1\pi + \beta_1 \sin 2s_1\pi.$$

This equation must not establish any relation between α_1 and β_1: hence we must have

$$\tan 2r\pi \cos\theta + \tan 2s\pi \sin\theta = \tan 2r_1\pi \ldots\ldots\ldots(3),$$

and $$-\tan 2r\pi \sin\theta + \tan 2s\pi \cos\theta = \tan 2s_1\pi \ldots\ldots\ldots(4).$$

The equations (1) and (2) are equivalent to

$$(+)^{-r}.x.\cos 2r\pi = (+)^{-r_1}.x_1.\cos 2r_1\pi.\cos\theta - (+)^{-s_1}.y_1.\cos 2s_1\pi.\sin\theta \ldots(5),$$
$$(+)^{-s}.y.\cos 2s\pi = (+)^{-r_1}.x_1.\cos 2r_1\pi.\sin\theta + (+)^{-s_1}.y_1.\cos 2s_1\pi.\cos\theta \ldots(6).$$

Thus (5) and (6) are the proper formulæ of transformation,

r_1 and s_1 being determinable, by the equations (3) and (4), in r, s, θ.

If $r = 0$, $s = 0$, then $r_1 = 0$, $s_1 = 0$, and the formulæ of transformation are reduced to

$$x = x_1 \cos\theta - y_1 \sin\theta, \quad y = x_1 \sin\theta + y_1 \cos\theta.$$

From the formulæ here given it is evident that neither the curvilinear loci nor the superficial locus of an equation between x and y is affected by the transformation of co-ordinates, and that accordingly no such objection, whatever may be its weight, as that urged by Mr. Salmon against Poncelet's method is available against Mr. Gregory's or mine.

I may here briefly remark, that the principles of the general interpretation of the equation $f(x, y) = 0$ by the doctrine of impossibles may be easily applied to the interpretation of the equation $f(x, y, z) = 0$. I may sometime enter more fully into this question.

Cambridge, March 17, 1852.

ON THE SIGNS + AND − IN GEOMETRY (*continued*), AND ON THE INTERPRETATION OF THE EQUATION OF A CURVE.

By Professor De Morgan.

Having recently had the advantage of looking over Mr. Salmon's two excellent works, the second edition of the *Conic Sections*, and that on the *Higher Plane Curves*, I beg to offer a few remarks on two points connected with the subject of those works: the general signification of an equation of two variables, first started, I believe, by D. F. Gregory; and the extension of the interpretation of the signs + and −, which I commenced in my *Differential Calculus*, and continued in this *Journal* (vol. vi. p. 156).

The first matter now stands thus: Gregory (*Camb. Math. Journal*, vol. i. p. 259) assigned to the equation $\phi(x, y) = 0$, the ordinary plane curve, and an infinity of curves not in the plane, derived from imaginary values of x and y; and from the first reading of his paper, I was satisfied that this infinity of curves constituted a surface. Mr. Walton (*Camb. Math. Journal*, vol. ii. p. 103, a paper which I did not know of until I saw Mr. Salmon's reference) maintains that the equation is always that of a surface, developes Gregory's mode of interpreting the imaginaries, and draws on the

surface an infinite number of *companions* to the plane curve. Mr. Salmon (*Higher Plane Curves*, p. 301) objects to the extension, in a manner which most readers of this *Journal* will see for themselves.

It seems to me that a distinction is required between algebraic geometry and geometrical algebra; using the substantive to denote the object-matter, the adjective to denote the auxiliary. We use algebra in aid of geometry to assist in gaining knowledge of forms: we use geometry in aid of algebra to assist in gaining representation of functions. In the first case, a circle, for instance, is our *datum*, and must not be associated with other curves at the bidding of algebra: in the second, a circle is the representation of the relations of real values in $x^2 + y^2 = \text{const.}$, and we are bound to select some extension which, while it includes this mode of representing the real values, shall also include more. The question, 'Given algebraical phenomena, to substitute geometrical ones for them,' is not answered until all relations of value are represented: and the geometer, in his search after the properties of defined curves, has no more right to limit the algebraist in his representation of defined functions, than the latter has a right to compel the former to enlarge his definition. Accordingly, I agree with Mr. Salmon in one capacity, and differ from him in the other.

The object of the *algebraist* is representation: and he adapts his coordinates accordingly. If, having to discuss $x^2 + y^2 = a^2$, he adopt rectilinear coordinates, but prefer polar ones for $y \cos x = a$, it is because each adaptation is the best for the case: he does not deny that, for some purposes, the character of the coordinates might be advantageously transposed. If a point be determined by three coordinates, and not more, it is because three is the *minimum* number; of which it is one convenience that each point has one set of coordinates only. But let there be $3 + m$ coordinates, then a surface will require $1 + m$ equations, and a curve $2 + m$. A curve, the geometrical intersection of surfaces, would indeed not be necessarily the result of all the equations of two surfaces, unless either $m = 0$, or the $2 + 2m$ equations be reducible to $2 + m$, m of them being deducible from the rest. When m is not $= 0$, a point on the common curve will generally be derived from different values of coordinates in the two surfaces.

Extending the number of coordinates, we preserve the Cartesian definition, namely, that each coordinate is in a line

of determinate direction chosen from among those which pass through a given point, and that the point determined by coordinates is at the extremity of their resultant. If there be six coordinates, $x, y, z, \xi, \eta, \zeta$, we must have four equations for a surface, say $V_1 = 0$, $V_2 = 0$, $V_3 = 0$, $V_4 = 0$. If we choose three axes of ordinary coordinates for ultimate reference, we have three equations of the form

$$X = ax + by + cz + \alpha\xi + \beta\eta + \gamma\zeta;$$

from which, with the four equations of the surface, we eliminate the six quantities x, ξ, &c., and produce an ordinary equation between X, Y, Z.

Generally, we can only assign two of the six x, ξ, &c.: but if we assign more than two values or relations, in such manner as to satisfy some of the four equations identically, we may thus be able to point out a curve upon the surface. Let, for instance, $\xi = 0$, $\eta = 0$, $\zeta = 0$, identically satisfy $V_2 = 0$, $V_4 = 0$; there remain two equations of the four, and the three $X = ax + by + cz$, &c., from which we may eliminate x, y, z, and thus produce two equations between X, Y, Z, belonging to a certain curve upon the surface. All this happens if we derive V_1 &c. from the two equations

$$\phi(x + \xi\sqrt{-1},\quad y + \eta\sqrt{-1},\quad z + \zeta\sqrt{-1}) = V_1 + V_2\sqrt{-1},$$
$$\psi(x + \xi\sqrt{-1},\quad y + \eta\sqrt{-1},\quad z + \zeta\sqrt{-1}) = V_3 + V_4\sqrt{-1}.$$

If then we take the ordinary curve

$$\phi(x, y, z) = 0,\quad \psi(x, y, z) = 0,$$

and ask how we may give representation to the solutions of these equations when imaginary values are used, one answer, among an infinite number, is as follows. Let the real parts of the coordinates be measured on the original axes, and the imaginary parts on three other axes chosen at pleasure through the same origin: let the point determined be at the extremity of the resultant of the six coordinates. Then the equations $\phi = 0$, $\psi = 0$, belong to a surface, and the curve of real values is on that surface. It does not follow that every surface can be so represented: for V_1 and V_2 are related functions, as are V_3 and V_4.

It will be observed that $\sqrt{-1}$ is not here assumed as the sign of perpendicularity: the three imaginary axes may be at any angles to the real axes and to one another. The attainment of full representation no more depends on x and ξ being at right angles to one another, than upon x and y being so related. It is not pretended that any

generalization of *operations* can take place: and, in so far as Mr. Gregory and Mr. Walton assume, if they do at all assume, that x and ξ must necessarily be at right angles to each other, I differ from them with Mr. Salmon, until further cause is shewn. With respect, however, to the transformation of coordinates, the case seems to stand as follows. This transformation can be made if we can assume three pairs of equations with the same coefficients, such as

$$x = ax_1 + by_1 + cz_1, \quad \xi = a\xi_1 + b\eta_1 + c\zeta_1;$$

x_1, ξ_1, &c. being the new coordinates. In this case, $x + \xi\sqrt{-1}$ becomes

$$a(x_1 + \xi_1\sqrt{-1}) + b(y_1 + \eta_1\sqrt{-1}) + c(z_1 + \zeta_1\sqrt{-1}),$$

and we clearly have, after substitution in $\phi = 0$, and $\psi = 0$, a pair of equations to the same absolute surface, with the same absolute curve in space for the real values. But this requires that a, b, c, &c., nine quantities in number, should satisfy twelve equations; that is, our supposition demands three relations between the twelve angles

$$xy,\ yz,\ zx, \quad x_1y_1, \text{ &c.}, \quad \xi\eta, \text{ &c.}, \quad \xi_1\eta_1, \text{ &c.}$$

The resultant of both sets of real coordinates is the same, and of both sets of imaginary coordinates. If the axes of x, y, z, make the same angles with one another which are made by the axes of ξ, η, ζ, and if the same be true of $x_1y_1z_1$, and $\xi_1\eta_1\zeta_1$, all conditions are satisfied; and the position of $x_1y_1z_1$ with respect to xyz is the same as that of $\xi_1\eta_1\zeta_1$ with respect to $\xi\eta\zeta$. But the permanence in space of the surface, under transformation of coordinates, concerns the algebraic geometer, not the geometrical algebraist.

The most symmetrical of all easy cases, is that in which the real axes of x, y, z, are the imaginary axes of ζ, ξ, η. Taking the axes of x, y, z, as the ultimate axes, we have then

$$X = x + \zeta, \quad Y = y + \xi, \quad Z = z + \eta.$$

If the real curve be $x^2 + y^2 = a^2$, $z = 0$, the surface is

$$(X - Z)(X^2 - Z^2 - a^2) + (X + Z)Y^2 = 0.$$

This surface is made by an hyperbola of variable minor axis revolving about the axis of z. The major semiaxis is always a: the minor varies from a to 0. The points requiring a moment's attention are the mode in which the axis of y belongs to the surface, and also the mode in which one asymptote of the equilateral hyperbola belongs to the surface at an infinite distance.

I am not sure I understand the particular way in which D. F. Gregory proposed to interpret the imaginary values. Mr. Walton makes the axis of z to be that of ξ and η both, so that the planes of xz and yz are separate planes of interpretation for x and y. This, whatever may be thought of it when $z + \zeta\sqrt{-1}$ also takes finite values, is the most striking method when the real curve is plane. It gives in that case

$$X = x, \quad Y = y, \quad Z = \xi + \eta.$$

And $x^2 + y^2 = a^2$ gives the surface

$$(X^2 + Y^2)\,Z^2 = (X^2 + Y^2 - a^2)\,(Y - X)^2.$$

The preceding *representations*, looked on as nothing more, should, I think, remain at one boundary of the subject, as indicating a direction in which further inquiry should be made. The particular cases in which one coordinate becomes imaginary, the other continuing real, are those which most want illustration, and which most decidedly receive it.

The second matter of this paper consists in some further applications of the method of interpreting + and − in plane geometry. They are no more than I have found to be necessary in making demonstrations coextensive with their enunciations. The numbering is continued from the paper already cited, and I have not thought it worth while to insert demonstrations. It is to be remembered that every equation here given is asserted to be universally true, under the previous conventions as to sign.

11. Let AB, BC, CA, be in the lines R, P, Q. Then

$$\begin{aligned} AC \sin Q^\circ R &= BC \sin P^\circ R = CB \sin R^\circ P, \\ AB &= AC \cos Q^\circ R + CB \cos R^\circ P, \\ AB^2 &= AC^2 + CB^2 + 2AC.CB \cos P^\circ Q, \\ &= AC^2 + BC^2 - 2AC.BC \cos P^\circ Q. \end{aligned}$$

12. If four concurrent lines A, B, C, D, contain the four colinear points a, b, c, d, then

$$\frac{ab}{bc} \cdot \frac{cd}{da} = \frac{\sin A^\circ B}{\sin B^\circ C} \cdot \frac{\sin C^\circ D}{\sin D^\circ A},$$

each side of which is − 1 when the pencil is harmonic.

13. Let every line drawn through the origin be called an *original*, and *the* original of all parallels to it. Again, every line divides the whole plane of reference into two half-planes. Let these half-planes take signs, that which contains the

origin taking sign from the original perpendicular *to* the straight line. But when the line is original, let its half-planes take sign from the parts of the original perpendicular with which the straight line makes a right angle. Inversion of the straight line is also inversion of the half-planes.

14. Parallels drawn through two points of the same original line have the same signature of translation, and make the same angles with any third line. And if P and Q be parallels of the same length and direction, the indefinite equation $P = \pm Q$ may be settled from the equation

$$P \sin P^{\circ}R = Q \sin Q^{\circ}R,$$

R being any third line: $P^{\circ}R$ and $Q^{\circ}R$ being either equivalents or *opponents*.*

15. Let the origin O be removed to $O'(m, n)$, and let new rectangular axes be taken, $x^{\circ}x$ being ϕ. Let OO' be of the same sign with reference to both the new axes, and let the new coordinates (measured *on*, not *parallel to*, their axes) take the signs determined by the system, $y'^{\circ}x'$ being $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$, as it shall happen. Then

$$x = m + x' \cos\phi - y' \sin\phi . \sin y'^{\circ}x',$$
$$y = n + x' \sin\phi + y' \cos\phi . \sin y'^{\circ}x',$$
$$x' = x \cos\phi + y \sin\phi - (m \cos\phi + n \sin\phi).$$

Let the axis of x' be original: and let $OO' = p$. Then

$$O'O = -(m \cos\phi + n \sin\phi),$$

and

$$x' = x \cos\phi + y \sin\phi - p.$$

That is to say, if the original perpendicular drawn *to* any straight line be p, and if $p^{\circ}x = \phi$, then the perpendicular drawn *from* the straight line to any point (x, y), projected upon the original perpendicular, is $x \cos\phi + y \sin\phi - p$: but the projected perpendicular drawn *to* the straight line is $p - x \cos\phi - y \sin\phi$; and this last has the sign of the half-plane in which (x, y) lies.

Let the above line be denoted by (p, ϕ): then $(p, \phi)^{\circ}x$ is always $\phi + \frac{1}{2}\pi$, and $(p, \phi)^{\circ}(q, \psi)$ is $(\phi + \frac{1}{2}\pi) - (\psi + \frac{1}{2}\pi)$ or $\phi - \psi$.

16. Two concurrent straight lines make four *Euclidean* angles, which may be named the $++$, $+-$, $-+$, $--$, angles after the signs of translation of the bounding lines drawn

* I call $\pi + \theta$ the *opponent* of θ, and $2\pi - \theta$ the *completion of* θ.

from the angular point. Each angular space is in one half-plane of each of its determining lines. And according as the signs of the angle are like or unlike, the signs of its half-planes are unlike or like.

17. If we give the name U to the line $U = 0$, and if $U = p - x\cos\phi - y\sin\phi$, then U, V, $aU + bV$, are concurrent straight lines, of which $aU + bV$ lies in the $++$ and $--$ angles of U and V when a and b have like signs, and in the $+-$ and $-+$ angles when they have unlike signs. Hence $U + V$ bisects the $++$ and $--$ angles of U, V, and $U - V$ bisects the $+-$ and $-+$ angles.

18. By the area of a closed figure is understood $\frac{1}{2}\int r^2 d\theta$, where $\theta = r^{\circ}x$, and r is the original distance of any point of the contour. This integral is to include all elements formed in making the circuit, the proper sign of $d\theta$ being taken for each, as in my paper (vol. v. p. 139) on the extension of the word *area*. The area is positive or negative, according to the way chosen of describing the contour. No easy universal rule can be given for distinguishing the positive and negative contours. But when the figure can be described entirely by positive revolution about a point within it, that mode of revolution gives the positive contour. The area is the same about any pole of revolution as about the origin, for one given mode of making the contour.

19. If R be the perpendicular let fall *from* a given point *to* a line in which lies AB, projected upon the original perpendicular to that line, the sign of $AB \times R$ is independent of the position of the origin. Provided only that when the origin is *on* the line of AB, the original perpendicular shall be that with which AB makes a right angle. And $\frac{1}{2}\Sigma(AB.R)$ is the area of any polygon, the sides being taken in order: and this area is positive when the order is (if such a thing can be) always that of positive revolution.

20. If $U = p - x\cos\phi - y\sin\phi$, &c., and if U, V, W intersect in A, B, C, then $BC.U + CA.V$ is a parallel to AB through the intersection of U and V.

21. If the four concurrent lines (p_1, ϕ_1), &c. have equations $a_1x + b_1y + c_1 = 0$, &c., the anharmonic ratio is

$$\frac{\sin(\phi_1 - \phi_2)}{\sin(\phi_2 - \phi_3)} \cdot \frac{\sin(\phi_3 - \phi_4)}{\sin(\phi_4 - \phi_1)} = \frac{\tan\phi_1 - \tan\phi_2}{\tan\phi_2 - \tan\phi_3} \cdot \frac{\tan\phi_3 - \tan\phi_4}{\tan\phi_4 - \tan\phi_1}$$

$$= \frac{a_1b_2 - a_2b_1}{a_2b_3 - a_3b_2} \cdot \frac{a_3b_4 - a_4b_3}{a_4b_1 - a_1b_4}.$$

The second formula does not change if all the tangents be augmented or multiplied by the same, or changed into their reciprocals, or subjected to any combination of these changes. One such combination is that which changes t into $(m + nt) : (m' + n't)$. Hence, U and V being any straight lines, the anharmonic ratio of $a_1 U + b_1 V$, &c. is the third of the preceding formulæ.

22. The radius of curvature is to be the line drawn *from* the centre of curvature *to* the curve. The formula $\frac{ds^3}{dx\,d^2y}$ always gives the projection of this radius on the original perpendicular to the tangent, if ds take sign on the tangent.

23. When a polygon is taken in the ordinary sense, all the original radii drawn to angular points must have one sign. Anything in contravention of this would amount to using a line and its inversion in determining two adjacent angles, not the same line in both.

24. Theorems relative to non-original lines are for convenience, not of necessity, often constructed with reference to their original parallels. Passage of a line through the origin either inverts that line, or its original radii: and theorems which are true of lines on one side of their original parallel, are often inverted for lines on the other.

I cannot find any instance in which the preceding extensions are either contradictory of each other, or insufficient either for demonstration or interpretation. And I think I can undertake to remove any difficulty in these respects, which they may present to any reader who will trouble himself to master their details.

March 25, 1852.

Though any case of the method given in the first part of this paper succeeds in representing all values of x and y, and illustrating all phenomena of curves, yet I am inclined to maintain that no one of them is the direct extension of the Cartesian method of coordinates. This direct extension I take to be as follows:

Let $X = (x, \xi)$ signify that X is a line of x units of length, inclined at the angle ξ to the unit-line. Then, Y being (y, η), X and Y are said to be *coordinates* of the other extremity of $X + Y$, when one extremity is the origin. That

is, X and Y are coordinates of a point whenever x, y, ξ, η are so taken as to satisfy two *equations of coordination*,

$$\phi(x, y, \xi, \eta) = 0, \quad \psi(x, y, \xi, \eta) = 0,$$

which define a *system* of coordinates. In the common system, the equations of coordination are $\xi = 0$, $\eta = \frac{1}{2}\pi$. If we take a third equation $F(x, y, \xi, \eta) = 0$,

we have the definition of a curve, of which this third may be called the equation. Any one of the four being given, the other three are determined. It can easily be shewn that even if we allow imaginary values for x, y, &c., we do not thereby gain any extension, but only conversion of one system of the above kind into another.

To this it will perhaps be objected that the third equation ought to be $F(X, Y) = 0$; for it will be said that an essential of the Cartesian method is the representation of the curve by a relation *between coordinates*. To this I reply that, when we import the Cartesian case into *complete* algebra, we do not find any equation between coordinates: for the coordinates are x and $y\sqrt{-1}$, while the equation is between x and y. What we do find is precisely that which I have *extended*, namely, three equations between the four *arithmetical* representatives of length and direction by which the two coordinates are determined. One equation between X and Y is equivalent to two equations between x, y, ξ, η. The matter which Gregory first opened, and on which Mr. Walton, Mr. Salmon, and myself, have hitherto written, is abandonment of a part of the Cartesian system, extension of the rest, and arbitrary introduction in place of what was abandoned. Coordination is abandoned: that is to say, all previous restriction on X and Y. The equation of the curve $F(x, y) = 0$, which was essentially a relation between lengths, independent of directions, is extended into $F(X, Y) = 0$, which gives two equations between $x\cos\xi$, $x\sin\xi$, $y\cos\eta$, $y\sin\eta$. And modes of laying down these last-named lengths are then chosen.

There is no question that the pure extension, as I take it to be, does not exhibit any method of interpreting the imaginary coordinates of the original system. For there are no such imaginary coordinates; the equations of coordination, $\xi = 0$, $\eta = \frac{1}{2}\pi$, cannot be disturbed in that system. The consideration of an equation between coordinates, in their most general form, requires abandonment, and is not compatible with pure extension or enlargement of existing modes of relation.

In the extension, the system of coordination

$$\phi(x, y, \xi, \eta) = 0, \quad \psi(x, y, \xi, \eta) = 0,$$

combined with the equation of the curve

$$F(x, y, \xi, \eta) = 0,$$

gives the ordinary curve obtained by eliminating x, y, ξ, η between the preceding equations and

$$x_1 = x \cos\xi + y \cos\eta, \quad y_1 = x \sin\xi + y \sin\eta.$$

July 5, 1852.

ON CERTAIN SYSTEMS IN SPACE ANALOGOUS TO THE COMPLETE TETRAGON AND COMPLETE QUADRILATERAL.*

By Thomas Weddle.

In a plane figure the straight line that joins two angles is called a *diagonal*, but so far as I know, no name has been bestowed on the point in which two sides intersect; such a point might, I think, be called a *collateral* with much propriety. (It will be observed that 'diagonal' and 'collateral' are reciprocal.) In Solid Geometry there are several analogous points, lines and planes, and I imagine the following terminology will be found convenient.

1. In a solid figure a *diagonal line* is a straight line joining two angular points not on the same edge.

2. A *diagonal plane* is a plane passing through three angular points not all in one face.

3. A *synhedral line* is the intersection of two faces that do not pass through the same edge.

4. A *synhedral point* is the point of intersection of three faces that do not meet in the same angular point.

5. When two edges intersect but are not in the same face, and do not meet in an angle, the point of intersection is called a *syngrammatic point*, or simply a *syngram*.

6. When two edges are in the same plane, but are not in the same face and do not meet in an angle, the plane passing through them is denominated a *diagrammatic plane*.

* This paper is intended to form a supplement to that part of my first memoir "On the Theorems in Space analogous to those of Pascal and Brianchon in a Plane," (*Journal*, new series, vol. IV. p. 26), which treats of umbilical surfaces.

It will be observed that, besides introducing several new terms, I have made an innovation by changing the signification of 'diagonal plane,' which is, I think, commonly used to denote what I have here denominated a 'diagrammatic plane'; but this use of the former term does not seem happy: the latter plane certainly passes through *angles* of the solid figure, but its characteristic is that it passes through *edges*, and it ought therefore if possible to be designated from this peculiarity; besides (and this with me is the strongest reason for the change) we want a name for the plane that passes through three angles, and *diagonal* plane from its etymology seems to be the appropriate term. It may perhaps be objected to the term 'diagrammatic' plane, that the word 'diagram' is used in another signification, but as I employ the adjective only, no confusion can, I think, arise, and the term may therefore be admissible.

Again, in Plane Geometry, if we name a figure from its sides, it is a *multilateral*; if from its angles, a *polygon*. In like manner, we name a solid figure a *polyhedron* from its faces; we may term it a *polygram** from its edges, and perhaps a *multangle* from its angular points. The last term is but little used in Plane Geometry except in the cases *triangle* and *quadrangle*, the former of which can never occur as the name of a solid figure, and the latter might be avoided in Plane Geometry by using *tetragon* instead.

Moreover, in Plane Geometry we have a *complete multilateral* formed by taking a number of straight lines and making each intersect all the others; and a *complete polygon* formed by taking a number of points and joining every two by straight lines. Now the most obvious analogies in Solid Geometry to these systems seem to be the following:

(1). A *complete polyhedron* is formed by taking a number (n) of planes for faces, and making every two planes intersect for edges and every three planes intersect for angles.

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* This term was suggested to me by the late T. S. Davies. *Perhaps* it may be convenient to use it in an extended sense so as to signify any system of straight lines in space.

The word 'polygram' has sometimes been used in the sense of 'multilateral' (*e.g.* Pascal's Hexagram), but so rarely as not to render its appropriation as above any inconvenience.

I am inclined to think that the terminology proposed in the text, down to 'multangle' inclusive, is appropriate and worthy perhaps of general adoption. The other terms serve my immediate object, but I am doubtful whether the study of figures in space is in a sufficiently advanced state to justify a decided opinion as to whether they will ultimately be found convenient or not.

Such a system, in its most general state, will have $\frac{n(n-1)}{1.2}$ edges and $\frac{n(n-1)(n-2)}{1.2.3}$ angular points.

(2). A *complete multangle* is formed by taking a number (n) of points in space for angles, joining every two of them by straight lines for edges and passing planes through every three for faces. Such a system, in its most general state, will have $\frac{n(n-1)}{1.2}$ edges and $\frac{n(n-1)(n-2)}{1.2.3}$ faces.

These definitions are, as I have already said, the most obvious analogies in space, but they exclude certain systems which, being analogous to the complete quadrilateral and complete tetragon, seem to have some title to the term 'complete.' To include these systems it will be necessary to modify the two preceding definitions, and after some consideration I am inclined to think that the best way of making the modification is by saying that in a *complete polyhedron* every point in which m of the faces intersect is an angular point, and every straight line in which two faces intersect and on which are at least $m-1$ angular points is an edge; and that in a *complete multangle* every plane passing through m angular points is a face, and every straight line which joins two angular points and through which at least $m-1$ faces pass is an edge. Whether these definitions will ultimately be found convenient may perhaps admit of some doubt,* but they suit my present purpose better than any others I can think of. It may be here observed that the only cases which I shall have to consider are $m=3$ (when these definitions coincide with the former two), and $m=4$.

Before proceeding further I must add two more definitions, the others will be given as wanted.

(1). If the opposite faces of a hexahedron intersect in three straight lines in one plane, the figure will have six diagrammatic planes and three syngrammatic points; such a solid figure I shall denominate a *syngrammatic octangular hexahedron*, or simply a *syngrammatic hexahedron*.

(2). If the straight lines joining the opposite angles of an octahedron intersect in a point, the figure will have three

* Would it be preferable to allow the former definitions of a complete polyhedron and complete multangle to stand, and to apply some other term than 'complete' to the systems that create the difficulty?

diagrammatic planes and six syngrammatic points. It would in some respects be convenient to call this solid a *diagrammatic octahedral sexangle;* but to depart as little as possible from the usual denomination, I shall term it a *diagrammatic sexangular octahedron*, or simply a *diagrammatic octahedron.*

It has been shewn in my first memoir "On the Theorems in Space analogous to those of Pascal and Brianchon in a Plane," (*Journal*, new series, vol. IV. p. 35), that the equations to the faces of a syngrammatic hexahedron may be denoted by

$$\left.\begin{array}{ll} T+S=0, & T-S=0 \\ U+S=0, & U-S=0 \\ V+S=0, & V-S=0 \end{array}\right\} \ldots\ldots\ldots (1),$$

the planes $T+S$ and $T-S$ being opposite faces, &c.; and that the equations to the diagrammatic planes are

$$\left.\begin{array}{ll} T+U=0, & T-U=0 \\ T+V=0, & T-V=0 \\ U+V=0, & U-V=0 \end{array}\right\} \ldots\ldots\ldots (2).$$

Let $ABCDA'B'C'D'$ be the angular points of the hexahedron, these points being denoted as follows:

$$\left.\begin{array}{llllll} A, & -S= & T= & U= & V; & A', \quad S= \quad T= \quad U= \quad V \\ B, & S=- & T= & U= & V; & B', -S=- \quad T= \quad U= \quad V \\ C, & S= & T=- & U= & V; & C', -S= \quad T=- \quad U= \quad V \\ D, & S= & T= & U=- & V; & D', -S= \quad T= \quad U=- \quad V \end{array}\right\} \ldots (3),$$

so that A is opposite to A', &c.

Again, let the diagonal straight lines joining opposite angular points intersect in E; also let the three sets of four edges each intersect in the points F, G, H; so that the points E, F, G, H, are denoted as follows:

$$\left.\begin{array}{l} E, \ T=U=V=0 \\ F, \ U=V=S=0 \\ G, \ V=S=T=0 \\ H, \ S=T=U=0 \end{array}\right\} \ldots\ldots\ldots\ldots (4).$$

The twelve planes (1) and (2) can be taken in four different ways, so that six of the planes shall be the faces of a syngrammatic hexahedron whose diagrammatic planes are the remaining six planes.

The planes (1) are the faces of one such hexahedron whose diagrammatic planes are the planes (2), whose opposite faces

intersect in the plane S, and whose syngrammatic points are F, G, and H.

Also the planes

$$S + T = 0, \quad S - T = 0,$$
$$U + T = 0, \quad U - T = 0,$$
$$V + T = 0, \quad V - T = 0,$$

are the faces of another syngrammatic hexahedron, whose diagrammatic planes are the other six planes, whose opposite faces intersect in the plane T, and whose syngrammatic points are G, H, and E.

Again, the planes

$$S + U = 0, \quad S - U = 0,$$
$$T + U = 0, \quad T - U = 0,$$
$$V + U = 0, \quad V - U = 0,$$

are the faces of a third syngrammatic hexahedron whose diagrammatic planes are the remaining six planes, whose opposite faces intersect in the plane U, and whose syngrammatic points are H, E, and F.

Finally, the planes

$$S + V = 0, \quad S - V = 0,$$
$$T + V = 0, \quad T - V = 0,$$
$$U + V = 0, \quad U - V = 0,$$

are the faces of a fourth syngrammatic hexahedron whose diagrammatic planes are the remaining six planes, whose opposite faces intersect in the plane V, and whose syngrammatic points are E, F, and G.

It will be observed that all these hexahedra have the same angular points $ABCDA'B'C'D'$.

Considering then (1) and (2) as a single system, it will have eight angular points $ABCDA'B'C'D'$, twelve faces (1, 2), sixteen edges and four syngrammatic points E, F, G, H; the faces pass six by six through the angles, six by six through the syngrams, and three by three through the edges. Such a system has much analogy to the complete tetragon, and I shall denominate it a *complete syngrammatic octangle*.

It is easily shewn that the points $ABCDEFGH$ are the angles of a complete syngrammatic octangle whose syngrams are $A'B'C'D'$; and that the points $A'B'C'D'EFGH$ are the angles of another complete syngrammatic octangle whose syngrams are $ABCD$. Hence any two of the three systems

of points $ABCD$, $A'B'C'D'$, $EFGH$, are the angles of a complete syngrammatic octangle of which the other four points are the syngrams.

A syngrammatic hexahedron may be viewed also as a complete system in another manner, as will appear further on.

Again, writing S for $2S$ in the equations at p. 32, vol. IV. new series, of this *Journal*, the faces of a diagrammatic octahedron may be denoted as follows:

$$\left.\begin{array}{ll} T+U+V+S=0, & T+U+V-S=0 \\ -T+U+V-S=0, & -T+U+V+S=0 \\ T-U+V-S=0, & T-U+V+S=0 \\ T+U-V-S=0, & T+U-V+S=0 \end{array}\right\} \ldots(5);$$

and the angles of this octahedron are easily found to be denoted as follows (they will be designated by the letters prefixed):

$$\left.\begin{array}{ll} a,\ U=V=T+S=0; & a',\ U=V=T-S=0 \\ b,\ T=V=U+S=0; & b',\ T=V=U-S=0 \\ c,\ T=U=V+S=0; & c',\ T=U=V-S=0 \end{array}\right\} ..(6);$$

and the syngrammatic points (which are all situated in the plane S), are

$$\left.\begin{array}{ll} \alpha,\ S=T=U+V=0; & \alpha',\ S=T=U-V=0 \\ \beta,\ S=U=T+V=0; & \beta',\ S=U=T-V=0 \\ \gamma,\ S=V=T+U=0; & \gamma',\ S=V=T-U=0 \end{array}\right\} ..(7).$$

The planes (5) may be taken opposite in four different ways, so as to form the faces of a diagrammatic octahedron. If the planes in the same horizontal line of (5) be considered opposite, then (5) will be the faces of a diagrammatic octahedron whose angular points are $aa'bb'cc'$, syngrammatic points $\alpha\alpha'\beta\beta'\gamma\gamma'$, diagrammatic planes T, U, and V, and whose opposite faces intersect in the plane S.

Again, taking the planes (5) opposite in the following manner,

$$\begin{array}{ll} U+V+S+T=0, & U+V+S-T=0, \\ -U+V+S-T=0, & -U+V+S+T=0, \\ U-V+S-T=0, & U-V+S+T=0, \\ U+V-S-T=0, & U+V-S+T=0, \end{array}$$

we have the faces of another diagrammatic octahedron whose angular points are $aa'\beta\beta'\gamma\gamma'$, syngrammatic points $\alpha\alpha'bb'cc'$, diagrammatic planes U, V, S, and whose opposite faces intersect in the plane T.

Also, taking the planes (5) opposite as follows,

$$\begin{aligned} V+S+T+U&=0, & V+S+T-U&=0,\\ -V+S+T-U&=0, & -V+S+T+U&=0,\\ V-S+T-U&=0, & V-S+T+U&=0,\\ V+S-T-U&=0, & V+S-T+U&=0, \end{aligned}$$

we get the faces of a diagrammatic octahedron whose angular points are $\alpha\alpha'bb'\gamma\gamma'$, syngrammatic points $aa'\beta\beta'cc'$, diagrammatic planes V, S, T, and whose opposite faces intersect in the plane U.

Finally, taking the planes (5) opposite in the following manner,

$$\begin{aligned} S+T+U+V&=0, & S+T+U-V&=0,\\ -S+T+U-V&=0, & -S+T+U+V&=0,\\ S-T+U-V&=0, & S-T+U+V&=0,\\ S+T-U-V&=0, & S+T-U+V&=0, \end{aligned}$$

we have the faces of a diagrammatic octahedron whose angular points are $\alpha\alpha'\beta\beta'cc'$, syngrammatic points $aa'bb'\gamma\gamma'$, diagrammatic planes S, T, U, and whose opposite faces intersect in the plane V.

Considering then the eight planes (5), and the twelve points (6, 7) as a single system, we shall have a system that has eight faces (5), twelve angular points (6, 7), sixteen edges, and four diagrammatic planes S, T, U, V; the angles lie six by six on the faces, six by six on the diagrammatic planes, and three by three on the edges. This system has considerable analogy to the complete quadrilateral, and I shall denominate it a *complete diagrammatic octahedron.*

It is not difficult to see that, of the three systems of planes

$$S=0,\quad T=0,\quad U=0,\quad V=0,$$

$$\left.\begin{aligned} -S+T+U+V&=0\\ S-T+U+V&=0\\ S+T-U+V&=0\\ S+T+U-V&=0 \end{aligned}\right\}$$

and

$$\left.\begin{aligned} S+T+U+V&=0\\ -S-T+U+V&=0\\ -S+T-U+V&=0\\ -S+T+U-V&=0 \end{aligned}\right\}$$

any two will be the faces of a complete diagrammatic octahedron of which the other four planes are the diagrammatic planes.

A diagrammatic octahedron may be viewed also as a complete system, in a different manner, as will be seen presently.

The general equation to surfaces of the second degree circumscribed about a syngrammatic hexahedron is (*Journal*, new series, vol. v. p. 66),

$$l(T+S)(T-S)+m(U+S)(U-S)+n(V+S)(V-S)=0,$$

or,
$$-(l+m+n)S^2+lT^2+mU^2+nV^2=0,$$

that is,
$$kS^2+lT^2+mU^2+nV^2=0$$

where
$$k+l+m+n=0 \quad \Big\} \ldots\ldots\ldots (8).$$

This is also of course the general equation to surfaces of the second degree circumscribed about (that is, passing though the angular points of) a complete syngrammatic octangle; and from (4) and (8) we infer the following theorem, which is perfectly analogous to a property of the complete tetragon.

I. *In a complete syngrammatic octangle, each of the four syngrammatic points is, relative to every surface of the second degree circumscribed about the octangle, the pole of the plane passing through the other three.*

Again (*Journal*, new series, vol. iv. p. 37), the general equation to surfaces of the second degree touching the faces (5) of a complete diagrammatic octahedron (recollecting that S must be written for $2S$), is

$$lT^2+mU^2+nV^2=S^2,$$

where
$$l^{-1}+m^{-1}+n^{-1}=1.$$

For symmetry write $-k^{-1}l$, $-k^{-1}m$, and $-k^{-1}n$, for l, m, and n, and the equation becomes

$$kS^2+lT^2+mU^2+nV^2=0$$

where
$$k^{-1}+l^{-1}+m^{-1}+n^{-1}=0 \quad \Big\} \ldots\ldots\ldots (9).$$

Hence, since S, T, U, and V are the diagrammatic planes, we have the following theorem, which is the reciprocal of (*I*) and is perfectly analogous to a property of the complete quadrilateral.

II. *In a complete diagrammatic octahedron, each of the four diagrammatic planes is, relative to every surface of the second degree inscribed in the octahedron, the polar of the intersection of the other three.*

Again (*ibid.* pp. 30, 35), the equations to the four surfaces of the second degree touching the edges of the four syn-

grammatic hexahedra which compose the complete syngrammatic octangle (1, 2) are

$$\left.\begin{aligned} T^2 + U^2 + V^2 &= 2S^2 \\ U^2 + V^2 + S^2 &= 2T^2 \\ V^2 + S^2 + T^2 &= 2U^2 \\ S^2 + T^2 + U^2 &= 2V^2 \end{aligned}\right\} \quad \ldots\ldots\ldots (10).$$

Hence

III. *In a complete syngrammatic octangle, each of the four syngrammatic points is, relative to the umbilical surface of the second degree touching the edges of any of the four syngrammatic hexahedra, the pole of the plane passing through the other three.*

Also (*ibid.* pp. 30, 36, recollecting to write S for $2S$ in equation (22), p. 30), the equations to the four surfaces of the second degree touching the edges of the four diagrammatic octahedra that can be formed out of the complete diagrammatic octahedron (5), are

$$\left.\begin{aligned} T^2 + U^2 + V^2 &= \tfrac{1}{2}S^2 \\ U^2 + V^2 + S^2 &= \tfrac{1}{2}T^2 \\ V^2 + S^2 + T^2 &= \tfrac{1}{2}U^2 \\ S^2 + T^2 + U^2 &= \tfrac{1}{2}V^2 \end{aligned}\right\} \quad \ldots\ldots\ldots (11).$$

Hence,

IV. *In a complete diagrammatic octahedron, each of the four diagrammatic planes is, relative to the umbilical surface of the second degree touching the edges of any of the four simple diagrammatic octahedra, the polar of the intersection of the other three.*

The general equation to surfaces of the second degree touching the faces of the syngrammatic hexahedron (1) is (*ibid.* p. 36)

$$\begin{aligned} &\sin^2\theta . T^2 + \sin^2\phi . U^2 + \sin^2\psi . V^2 + 2\cos\theta . \sin\phi . \sin\psi . UV \\ &\quad + 2\cos\phi . \sin\psi . \sin\theta . TV + 2\cos\psi \sin\theta \sin\phi . TU \\ &\quad = (1 - \cos^2\theta - \cos^2\phi - \cos^2\psi + 2\cos\theta . \cos\phi . \cos\psi)\, S^2 \ldots (12), \end{aligned}$$

so that the point $T = U = V = 0$ is the pole of the plane S. Hence,

V. *In a syngrammatic hexahedron, the point in which the diagonal straight lines joining opposite angles intersect is, relative to any surface of the second degree inscribed in the hexahedron, the pole of the plane in which the opposite faces intersect.*

Also, if $t = 0$, $u = 0$, $v = 0$, $w = 0$, $t' = 0$, $u' = 0$, $v' = 0$, $w' = 0$, be the equations to the faces of any octahedron, t and t' being opposite, &c., then (*Journal*, new series, vol. v. p. 66, foot-

note), the general equation to surfaces of the second degree circumscribed about the octahedron will be

$$ftt' + guu' + hvv' + kww' = 0.$$

When the octahedron is diagrammatic, we shall have

$$t = T + U + V + S, \quad t' = T + U + V - S, \text{ \&c.},$$

as in (5); and then the last equation reduces to

$$T^2 + U^2 + V^2 + 2lUV + 2mTV + 2nTU = S^2 \ldots (13),$$

where $$l = \frac{f + g - h - k}{f + g + h + k}, \text{ \&c.}$$

Hence the point $T = U = V = 0$ being the pole of the plane $S = 0$, we see that

VI. *In a diagrammatic octahedron the point in which the diagonal straight lines intersect is, relative to every surface of the second degree circumscribed about the octahedron, the pole of the plane in which the opposite faces intersect.*

In order that the products UV, VT, TU should disappear from (12) and that the other terms should not disappear, we must have $\cos\theta = \cos\phi = \cos\psi = 0$, giving $\sin^2\theta = \sin^2\phi = \sin^2\psi = 1$, when the equation reduces to $T^2 + U^2 + V^2 = S^2$; now it will be found that this surface (which is evidently umbilical) touches each face in the point in which the diagonals of that face intersect; and it may easily be shewn conversely, that if a surface of the second degree touch the faces at these points, its equation will reduce to the preceding form. Hence the equations to the four surfaces touching (in the manner just mentioned) the faces of the four syngrammatic hexahedra which compose the complete syngrammatic octangle, are

$$\left.\begin{aligned} T^2 + U^2 + V^2 &= S^2 \\ U^2 + V^2 + S^2 &= T^2 \\ V^2 + S^2 + T^2 &= U^2 \\ S^2 + T^2 + U^2 &= V^2 \end{aligned}\right\} \ldots\ldots\ldots (14).$$

Hence,

VII. *In a complete syngrammatic octangle, each of the four syngrammatic points is the pole of the plane passing through the other three, relative to the umbilical surface of the second degree touching the faces of any of the four syngrammatic hexahedra at the intersections of the diagonals of these faces.*

Also, that the products UV, VT, TU should disappear from (13), we must have $l = m = n = 0$, and it will then be

found that the surface touches the twelve synhedral lines that pass through the angles of the octahedron; thus the umbilical surface $T^2 + U^2 + V^2 = S^2$ touches the synhedral line in which the faces $-T+U+V-S=0$ and $T-U+V-S=0$ intersect. Conversely, it is easy to shew that if a surface of the second degree circumscribed about the octahedron touch these twelve lines, its equation must be of the preceding form. Hence (14) are the equations to the four umbilical surfaces circumscribed about the four simple diagrammatic octahedra that can be formed by the faces of a complete diagrammatic octahedron, these surfaces moreover touching the synhedral lines that pass through the angles. Hence,

VIII. *In a complete diagrammatic octahedron each of the diagrammatic planes is the polar of the intersection of the other three, relative to the umbilical surface of the second degree which is circumscribed about any of the four simple diagrammatic octahedra, and which at the same time touches the twelve synhedral lines that pass through its angles.*

Other theorems somewhat similar to those already given might be noticed, but as they are of less interest I shall omit them.

The twelve faces (1, 2) of a complete syngrammatic octangle form six pairs, and the planes of each pair intersect in an edge of the tetrahedron whose angles coincide with the syngrammatic points (so that the equations to the faces of the tetrahedron are $S=0$, $T=0$, $U=0$, $V=0$). Thus the two faces $U+V=0$ and $U-V=0$ form one pair, and these intersect in the straight line $U=V=0$, which is one of the edges of the tetrahedron. Also it will be observed that the four planes U, V, $U-V$, and $U+V$ form a harmonic system;* hence the following theorem:

IX. *The twelve faces of a complete syngrammatic octangle form six pairs, the planes of each pair intersecting in an edge of the tetrahedron whose angles are at the four syngrammatic points; also the two faces of the tetrahedron and the two faces of the octangle that intersect in any edge of the tetrahedron, form a harmonic system.*

Again, in a complete diagrammatic octahedron, the twelve angular points form six pairs (aa', bb', cc', $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$), the two points of each pair being situated on an edge of the tetrahedron whose faces are the four diagrammatic planes S, T, U, V.

* When I say that four points or planes form a harmonic system, I mean that the first two are harmonic conjugates with respect to the last two.

Thus, (6), the points aa' are situated on the edge $U=V=0$ of the tetrahedron; also since the planes $T+S$, $T-S$, T and S (which intersect in the line $S=T=0$) pass through the two points a, a', and the two angles of the tetrahedron on the edge $U=V=0$, it follows that these four points form a harmonic system.

X. *The twelve angular points of a complete diagrammatic octahedron form six pairs, each pair of points being situated on an edge of the tetrahedron whose faces coincide with the four diagrammatic planes; also any pair of points with the two angular points of the tetrahedron that are on the same edge form a harmonic system.*

Since S, T, $T-S$, $T+S$, and $U\pm S$, $\pm V\pm S$, $U\mp V\{=(U\pm S) - (\pm V\pm S)\}$, $U\pm V\pm 2S\{=(U\pm S)+(\pm V\pm S)\}$, in the hexahedron are harmonic systems; S, $T\pm U\pm V$, $(T\pm U\pm V)-S$, $(T\pm U\pm V)+S$, and T, $S\pm U\pm V$, $(S\pm U\pm V)-T$, $(S\pm U\pm V)+T$, in the octahedron are harmonic systems (here the double signs are independent before different letters but not before the same letter), we infer, remembering that $U\pm V\pm 2S$, and $S\pm U\pm V$ are tangent planes to the hexahedron and octahedron respectively (see the paper before referred to), that

XI. *If an umbilical surface of the second degree touch the edges of either a syngrammatic hexahedron or diagrammatic octahedron, the two angular points, the syngrammatic point and the point of contact, on any edge, form a harmonic system; also the two faces, the diagrammatic plane and the tangent plane, passing through any edge, form another harmonic system.**

Other theorems, somewhat similar to the last two or three, might be given, but I hasten to another part of my subject.

A pentangular† *hexahedron* is a solid figure generated by taking two tetrahedra of equal bases and making these bases coincide; or it may be formed by taking five points $AA'FGH$ in space and joining every two of them by straight lines except one pair AA'. The figure will thus have nine edges, and one diagonal line AA'; there are four diagonal planes, AFA', AGA', AHA', and FGH, the last of which may be termed the *unique* diagonal plane, and the first three pass through the diagonal line AA', so that the four diagonal planes intersect in a point situated on the diagonal line. The

* The theorems (XI.) enable us to find the points of contact on, and the tangent planes through, the edges, when the solid alone is given.

† I trust the reader will excuse my using 'pentangular' instead of the much less euphonious term 'quinquangular,' notwithstanding that the latter is etymologically the more correct.

nine edges may be divided into three sets—a convergent set AF, AG, AH intersecting in A, a second convergent set $A'F$, $A'G$, $A'H$ intersecting in A', and a non-convergent set GH, HF, FG situated in the unique diagonal plane. Also the first, second, and third of each set may be denominated *corresponding* edges, and each pair of planes AGH, $A'GH$; AHF, $A'HF$; AFG, $A'FG$, *corresponding* faces.

Let $t=0$, $u=0$, $v=0$ denote the faces AGH, AHF, and AFG respectively; also let $S=0$ denote the unique diagonal plane: supposing t, u, and v to have been multiplied by the proper constants, the corresponding faces may be denoted by

$$t-2S=0, \quad u-2S=0, \quad v-2S=0.$$

Substituting $T+S$, $U+S$, and $V+S$ for t, u, and v, we see that the faces of a pentangular hexahedron may be denoted as follows:

$$\left.\begin{aligned} T+S=0, \quad & T-S=0 \\ U+S=0, \quad & U-S=0 \\ V+S=0, \quad & V-S=0 \end{aligned}\right\} \ldots\ldots\ldots (15),$$

in which the equations to the corresponding faces are placed in the same horizontal line. The intersections of the corresponding faces form the non-convergent edges, the mutual intersections of the three planes on the left one set of convergent edges, and the mutual intersections of the three planes on the right form the other set of convergent edges.

It is evident that the equation to the unique diagonal plane is $S=0$, the equations to the other three diagonal planes

$$U-V=0, \quad V-T=0, \quad T-U=0,$$

and those of the diagonal line $T=U=V$.

The equation to the surface of the second degree touching the edges of the pentangular hexahedron is easily seen to be

$$4S^2+T^2+U^2+V^2-2UV-2TV-2TU=0 \ldots (16);$$

and since this equation may be put under the form

$$4S^2+(-T+U+V)^2+(U-V)^2=(U+V)^2,$$

the surface is umbilical.

From (16) it is easily shewn that the equations to the straight lines joining the points of contact of the corresponding edges of the two convergent sets, are

$$\left.\begin{aligned} \tfrac{1}{2}T=U=V \\ \tfrac{1}{2}U=T=V \\ \tfrac{1}{2}V=T=U \end{aligned}\right\} \ldots\ldots\ldots\ldots (17),$$

and these intersect in the point $T=U=V=0$ on the diagonal line. Let E be this point.

The equations to the straight lines joining the points of contact of the corresponding edges of the non-convergent set and the convergent set intersecting in A, are

$$\left.\begin{aligned} T+2S=U-V=0 \\ U+2S=V-T=0 \\ V+2S=T-U=0 \end{aligned}\right\} \ldots\ldots\ldots (18),$$

and these intersect in the point $T=U=V=-2S$ on the diagonal line.

Finally, the equations to the straight lines joining the points of contact of the corresponding edges of the non-convergent set and the convergent set intersecting in A', are

$$\left.\begin{aligned} T-2S=U-V=0 \\ U-2S=V-T=0 \\ V-2S=T-U=0 \end{aligned}\right\} \ldots\ldots\ldots (19),$$

and these intersect in the point $T=U=V=2S$ on the diagonal line.

It is evident that the corresponding lines of (17), (18), and (19) are situated in the diagonal planes $U-V=0$, $V-T=0$, and $T-U=0$ respectively, and hence these planes intersect the edges in the unique diagonal plane in their points of contact.

Let the diagonal line and the unique diagonal plane intersect in L. Since the planes S, T, $T+S$, and $T-S$ pass through the points L, E, A, A' on the diagonal line, these points form a harmonic system.

It is at once seen from (16) that

$$T=0, \quad U=0, \quad V=0 \ldots\ldots\ldots (20)$$

are the equations to the tangent planes passing through the edges GH, HF, and FG in the unique diagonal plane, and these planes intersect on the diagonal line in the same point (E) as the lines (17).

The equations to the tangent planes through the other edges are

$$\left.\begin{aligned} U+V+2S=0 \\ V+T+2S=0 \\ T+U+2S=0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (21),$$

and

$$\left.\begin{aligned} U+V-2S=0 \\ V+T-2S=0 \\ T+U-2S=0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (22).$$

Since each set of four planes $T+S$, $T-S$, S, T, and $U \pm S$, $V \pm S$, $U-V\{=(U \pm S)-(V \pm S)\}$, $U+V \pm 2S\{=(U \pm S)+(V \pm S)\}$, forms a harmonic system, it follows that the two faces, the diagonal plane, and the tangent plane, that pass through any edge, form a harmonic system.

Again, since each set of four planes S, $T+S$, $T\{(=T+S)-S\}$, $T+2S\{=(T+S)+S\}$, and S, $T-S$, $T\{=(T-S)+S\}$, $T-2S\{=(T-S)-S\}$ forms a harmonic system, it is easy to infer that the two angles on any convergent edge, its point of contact, and the point in which the said edge is intersected by the tangent plane through the corresponding non-convergent edge, form a harmonic system.

It is evident that the point E or $T=U=V=0$ is the pole, relative to (16), of the unique diagonal plane. Hence

XII. *An umbilical surface of the second degree can be drawn to touch the edges of a pentangular hexahedron; the three diagonal planes intersect the non-convergent edges in their points of contact; the three straight lines joining the points of contact of the corresponding edges of any two of the three sets intersect in a point on the diagonal line; the tangent planes to the surface drawn through the non-convergent edges intersect in that point (E) on the diagonal line in which intersect the three lines joining the points of contact of the corresponding edges of the two convergent sets; and this point is the pole, relative to the surface, of the unique diagonal plane. The two faces, the diagonal plane, and the tangent plane, that pass through any edge, form a harmonic system; the two angles on any convergent edge, its point of contact, and the point in which the said edge is intersected by the tangent plane through the corresponding non-convergent edge, form a second; and the two angles that are on the diagonal line, the point of intersection of the diagonal planes, and the point E, form a third harmonic system.**

Since the equations (15) coincide with the equations (1), it is evident that (12) will be the general equation to surfaces of the second degree inscribed in the pentangular hexahedron; hence also (*Journal*, new series, vol. IV. p. 37) the equations to the straight lines joining the points of contact of corresponding faces are

$$\left.\begin{aligned} -T &= lU = kV \\ -U &= lT = hV \\ -V &= kT = hU \end{aligned}\right\} \ldots\ldots\ldots\ldots (23),$$

* By aid of some of the properties (XII.) we can find the tangent planes through, and the points of contact on, the edges, when the solid alone is given.

where $h = \dfrac{\sin\phi \sin\psi}{\cos\theta - \cos\phi \cos\psi}$, &c., and these always intersect in the fixed point E or $T = U = V = 0$ on the diagonal line, which point is, (12), the pole of the unique diagonal plane. Hence

XIII. *If any surface of the second degree be inscribed in a pentangular hexahedron, the three straight lines joining the points of contact of corresponding faces intersect in a point on the diagonal line. This point is the same for all inscribed surfaces, and is, relative to any of them, the pole of the unique diagonal plane.*

Since the equations (15) coincide with the equations (1), we see that the faces of a pentangular hexahedron, and those of a syngrammatic octangular hexahedron, form the very same system of planes, one of the chief differences in the two solid figures being that in the first the corresponding faces are *adjacent*, while in the latter they are *opposite*. Indeed, if the three straight lines in which three pairs of planes intersect be in one plane, these six planes can be taken so as to form the faces of one syngrammatic hexahedron and four pentangular hexahedra.

The equations of such a system of planes being denoted by

$$\left.\begin{array}{ll} T + S = 0, & T - S = 0 \\ U + S = 0, & U - S = 0 \\ V + S = 0, & V - S = 0 \end{array}\right\} \ldots\ldots\ldots (24),$$

if we take the corresponding planes (those in the same horizontal line) for opposite faces, we shall have a syngrammatic octangular hexahedron, but if we take them as adjacent faces we may form a pentangular hexahedron in four different ways. If (besides making the corresponding planes of (24) intersect) we make the three planes on the left intersect two and two, as also the three planes on the right, we shall have one of the pentangular hexahedra.

Again, writing the preceding equations in the order

$$\left.\begin{array}{ll} T - S = 0, & T + S = 0 \\ U + S = 0, & U - S = 0 \\ V + S = 0, & V - S = 0 \end{array}\right\} \ldots\ldots\ldots (25),$$

and making the planes on the right and left intersect two and two as before, we shall have another pentangular hexahedron.

By arranging the planes as follows,

$$\left.\begin{aligned} &T+S=0, \quad T-S=0\\ &U-S=0, \quad U+S=0\\ &V+S=0, \quad V-S=0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (26),$$

and

$$\left.\begin{aligned} &T+S=0, \quad T-S=0\\ &U+S=0, \quad U-S=0\\ &V-S=0, \quad V+S=0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (27),$$

we shall get other two pentangular hexahedra. Denoting the angular points by the letters in (3) and (4), the angular points of the syngrammatic octangular hexahedron are $ABCDA'B'C'D'$, and those of the pentangular hexahedra $AA'FGH$, $BB'FGH$, $CC'FGH$, and $DD'FGH$, respectively. Hence the pentangular hexahedra have three angular points F, G, H, and one diagonal plane FGH (or $S=0$) common; also their other angles coincide with opposite angles of the octangular hexahedron, so that the diagonal lines AA', BB', CC', DD' of the pentangular hexahedra are the diagonal lines joining opposite angles of the octangular hexahedron. The entire system thus consists of six faces, fifteen edges, and eleven angles; it has a good deal of analogy to the complete quadrilateral, and may be denominated a *complete undecangular* hexahedron.*

The equation to the surface of the second degree touching the edges of the octangular hexahedron is of course

$$T^2+U^2+V^2=2S^2 \ldots\ldots\ldots\ldots\ldots\ldots (28);$$

also the equations to the surfaces of the second degree touching the edges of the pentangular hexahedra (taken in the preceding order), are

$$\left.\begin{aligned} &4S^2+T^2+U^2+V^2-2UV-2TV-2TU=0\\ &4S^2+T^2+U^2+V^2-2UV+2TV+2TU=0\\ &4S^2+T^2+U^2+V^2+2UV-2TV+2TU=0\\ &4S^2+T^2+U^2+V^2+2UV+2TV-2TU=0 \end{aligned}\right\} \ldots (29).$$

Equation (12) is of course the general equation to surfaces of the second degree touching the faces of the complete undecangular hexahedron, and (8) the general equation to

* The adjective 'undecangular' is added to distinguish the system from the complete hexahedron *in its general state*, which has *twenty* angular points.

surfaces of the second degree circumscribed about the octangular hexahedron. The general equation to surfaces of the second degree circumscribed about each of the pentangular hexahedra might also be given, but it would be of little utility.

It is easily seen that such of the surfaces (28) and (29) as touch the same edge have the same tangent plane passing through that edge; the equations to the tangent planes passing through the various edges are as follows:

$$\left.\begin{array}{ll} GH, & T=0 \\ HF, & U=0 \\ FG, & V=0 \\ AB', & U+V+2S=0 \\ A'B, & U+V-2S=0 \\ AC', & T+V+2S=0 \\ A'C, & T+V-2S=0 \\ AD', & T+U+2S=0 \\ A'D, & T+U-2S=0 \\ BC', & T-U+2S=0 \\ B'C, & -T+U+2S=0 \\ BD', & T-V+2S=0 \\ B'D, & -T+V+2S=0 \\ CD', & U-V+2S=0 \\ C'D, & -U+V+2S=0 \end{array}\right\} \quad \ldots\ldots\ldots (30).$$

There are many interesting properties of this system of surfaces, their points of contact with the edges of the complete undecangular hexahedron, &c., but I do not deem it advisable to enlarge upon these topics. Most of the preceding theorems that relate to hexahedra of any kind may be enunciated as properties of this system, but the following will suffice.

XIV. *In a complete undecangular hexahedron, the point in which the four diagonal lines of the pentangular hexahedra intersect is the pole of the diagonal plane common to the same, with respect to any of the following surfaces of the second degree—any of the five surfaces touching the edges of the five simple hexahedra, any surface inscribed in the system, or any surface circumscribed about the octangular hexahedron. The three straight lines joining the points of contact of corresponding faces with any inscribed surface of the second degree pass through the point first mentioned; and (omitting the three edges common to all the pentangular hexahedra) the straight lines joining*

the corresponding points of contact of the five surfaces touching edges intersect in the same point; also (without any omission) such of these five surfaces as touch any edge have a common tangent plane passing through that edge.

A sexangular pentahedron is a solid figure generated by taking a tetrahedron and cutting off a portion towards one of its angles by a plane; or, it may be formed by taking any five planes and making every two of them intersect except one pair.* The figure will thus have three quadrilateral and two triangular faces; let a, b, c denote the angles of one of the triangles, and a', b', c' those of the other, aa' being on the same edge, &c., so that a and a', b and b', c and c' will be *corresponding* angles. There is one synhedral line, that in which the planes abc, $a'b'c'$ intersect, and four synhedral points, three of which are situated on the synhedral line, and the other (which may be called the *unique* synhedral point) is the intersection O of the three quadrilateral faces, so that the four synhedral points are situated in one plane. The edges consist of three sets; one convergent set aa', bb', cc', that intersect in the unique synhedral point, and two non-convergent sets bc, ca, ab; $b'c', c'a', a'b'$, which are the edges in the two triangular faces; also each of the following triads, $aa', bc, b'c'$; $bb', ca, c'a'$; $cc', ab, a'b'$, may be called *corresponding* edges.

Let $T=0$, $U=0$, $V=0$, denote the quadrilateral faces, and $s=0$, the triangular face abc; supposing T, U, V to have been multiplied by the proper constants, the equation to the triangular face $a'b'c'$ may be denoted by $2T+2U+2V-s=0$, or substituting $T+U+V+S$ for s, the equations to the faces of a sexangular pentahedron may be denoted as follows:

$$\left.\begin{array}{l} T=0, \quad U=0, \quad V=0, \\ T+U+V+S=0, \quad T+U+V-S=0 \end{array}\right\} \ldots\ldots (31),$$

the first three denoting the quadrilateral faces. The equations to the synhedral line are evidently $T+U+V=S=0$; those of the unique synhedral point (O), $T=U=V=0$; and that of the plane passing through the four synhedral points, that is, through the synhedral line and unique synhedral point, $T+U+V=0$.

The equation to the surface of the second degree, touching the nine edges of the sexangular pentahedron, is readily seen to be

$$S^2 = 4UV + 4TV + 4TU \ldots\ldots\ldots (32);$$

* It is evident that the pentangular hexahedron and the sexangular pentahedron are reciprocal figures.

and since this equation may be written in the form

$$S^2 + 4T^2 + (U - V)^2 = (2T + U + V)^2,$$

the surface is umbilical.

We see at a glance that $S = 0$ ..... (33)

is the equation to the plane passing through the points of contact of the convergent edges, and it evidently passes through the synhedral line. We also perceive that this plane, the plane $T + U + V$ of the synhedral points, and the two triangular faces $T + U + V + S$, $T + U + V - S$, form a harmonic system.

The equations to the three straight lines joining the points of contact of the corresponding edges of the two non-convergent sets, are

$$\left.\begin{aligned} T = U - V = 0 \\ U = V - T = 0 \\ V = T - U = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (34),$$

which intersect in the unique synhedral point O or $T = U = V = 0$.

The equations to the lines joining the points of contact of the corresponding edges of the convergent set and the set in the face $T + U + V + S$, are

$$\left.\begin{aligned} U = V = -\tfrac{1}{2}S \\ V = T = -\tfrac{1}{2}S \\ T = U = -\tfrac{1}{2}S \end{aligned}\right\} \ldots\ldots\ldots\ldots (35),$$

which intersect in the point $T = U = V = -\frac{1}{2}S$. Denote this point by n.

Finally, the equations to the three lines joining the points of contact of the corresponding edges of the convergent set and the set in the face $T + U + V - S = 0$, are

$$\left.\begin{aligned} U = V = \tfrac{1}{2}S \\ V = T = \tfrac{1}{2}S \\ T = U = \tfrac{1}{2}S \end{aligned}\right\} \ldots\ldots\ldots\ldots (36);$$

and these pass through the point $T = U = V = \frac{1}{2}S$. Call this point n'.

It is evident that the three points O, n, n', are on the straight line $T = U = V$.

It is plain that the corresponding lines of (34), (35), and (36) are situated in the planes

$$\left.\begin{aligned} U - V = 0 \\ V - T = 0 \\ T - U = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (37),$$

respectively; these planes being, in fact, the planes that are drawn through the convergent edges and the points of contact of the corresponding non-convergent edges.

The equations to the three planes, each of which is drawn through one of the convergent edges and the corresponding synhedral point, are

$$\left.\begin{aligned} U+V&=0\\ V+T&=0\\ T+U&=0 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(38);$$

and these are also the equations to those tangent planes of (32) that are drawn through the convergent edges.

Since $T+U+V+S$, $T+U+V-S$, $T+U+V$, S, and U, V, $U+V$, $U-V$, each form a harmonic system, we see that the two angles, the synhedral point and the point of contact of any edge, form a harmonic system.

The equations to the tangent planes of (32) passing through the non-convergent edges in the faces $T+U+V+S$ and $T+U+V-S$ respectively, are

$$\left.\begin{aligned} 2T+\ U+\ V+S&=0\\ T+2U+\ V+S&=0\\ T+\ U+2V+S&=0 \end{aligned}\right\}\ldots\ldots\ldots(39),$$

and

$$\left.\begin{aligned} 2T+\ U+\ V-S&=0\\ T+2U+\ V-S&=0\\ T+\ U+2V-S&=0 \end{aligned}\right\}\ldots\ldots\ldots(40).$$

Since T, $T+U+V\pm S(=\Sigma$ suppose), $2T+U+V\pm S(=\Sigma+T)$, $U+V\pm S(=\Sigma-T)$, and U, V, $U+V$, $U-V$, each form a harmonic system, we see that the two faces and the tangent plane through any edge, together with the plane passing through the said edge and the point of contact of the corresponding edge, form a harmonic system.

It is evident, both from (32) and geometrical considerations, that the synhedral point O or $T=U=V=0$, is the pole, relative to (32), of the plane $S=0$.

XV. *An umbilical surface of the second degree can be drawn to touch the edges of a sexangular pentahedron. The three straight lines joining the points of contact of the corresponding edges of any two of the three sets intersect in a point, and the three points (one of which is the unique synhedral point) thus found are in a straight line. Each of the tangent planes to the surface drawn through the convergent edges, which of course passes through the unique synhedral point, likewise passes through one of the other synhedral points. The plane (S) of the points of contact of the*

*convergent edges passes through the synhedral line; and the unique synhedral point is the pole, relative to the surface, of this plane. The two angles, the synhedral point and the point of contact, of any edge, form a harmonic system; the two triangular faces, the plane of the synhedral points, and the plane S, a second; and the two faces and the tangent plane through any edge, together with the plane passing through the said edge, and the point of contact of the corresponding edge, form a third harmonic system.**

It is easily shewn from (31) that the equations to the angular points of the sexangular pentahedron are denoted as in (6); hence these points will be the angles of a diagrammatic octahedron, whose faces are denoted by (5). Consequently (13) is the general equation to surfaces of the second degree circumscribed about the sexangular pentahedron. The equations to the tangent planes touching (13) at the angles of the pentahedron, will be found to be

$$\left.\begin{array}{ll} mV+nU+T+S=0, & mV+nU+T-S=0 \\ lV+nT+U+S=0, & lV+nT+U-S=0 \\ lU+mT+V+S=0, & lU+mT+V-S=0 \end{array}\right\} \ldots(41);$$

and the opposite planes always intersect in the fixed plane $S=0$; also it is evident that this plane is, relative to (13), the polar of the unique synhedral point $T=U=V=0$.

XVI. *If any surface of the second degree be circumscribed about a sexangular pentahedron, the tangent planes at the corresponding angles will intersect in three straight lines in a plane passing through the synhedral line. This plane is the same for all circumscribed surfaces, and is, relative to any of them, the polar of the unique synhedral point.*

Since (as has been observed) the equations (5) to the faces of a diagrammatic octahedron, and the equations (31) to those of a sexangular pentahedron, both give the equations (6) for the angles, we see that the angular points of both solids form precisely the same system, one of the chief differences being, that in the octahedron the corresponding angles are *opposite*, while in the pentahedron they are *adjacent*, that is, are on the same edge. Indeed, if the three straight lines joining three pairs of points pass through the same point, these six points can be taken so as to form the angles of one diagrammatic octahedron and of four sexangular pentahedra.

It has been virtually shown in my first memoir (repeatedly referred to) on the "Theorems in Space analogous to those

* Hence we can find the points of contact on, and the tangent planes through, the edges, when the solid alone is given.

of Pascal and Brianchon in a Plane," that under these circumstances the faces of the octahedron may be denoted by (5), and hence the six points may be denoted by (6).

The points a and a', b and b', c and c', being taken for opposite angular points, they will form the angles of a diagrammatic octahedron whose faces are denoted by (5), and whose diagrammatic planes are T, U, and V.

But if we take a and a', b and b', c and c', as adjacent angles (that is, on the same edges), then these points may be taken as the angles of a sexangular pentahedron in four different ways.

If the triangular faces be abc, $a'b'c'$, we shall get one pentahedron whose faces are

$$\left.\begin{aligned} T=0,\ \ U=0,\ \ V=0, \\ T+U+V+S=0,\ \ T+U+V-S=0, \end{aligned}\right\} \ldots(42).$$

Again, if the triangular faces be $ab'c'$, $a'bc$, the faces of a second pentahedron will be

$$\left.\begin{aligned} T=0,\ \ U=0,\ \ V=0, \\ -T+U+V-S=0,\ \ -T+U+V+S=0, \end{aligned}\right\} \ldots(43).$$

Thirdly, if the triangular faces be $a'bc'$, $ab'c$, the faces of the third pentahedron will be

$$\left.\begin{aligned} T=0,\ \ U=0,\ \ V=0, \\ T-U+V-S=0,\ \ T-U+V+S=0, \end{aligned}\right\} \ldots(44).$$

Finally, if the triangular faces be $a'b'c$, abc', the faces of the fourth pentahedron will be

$$\left.\begin{aligned} T=0,\ \ U=0,\ \ V=0, \\ T+U-V-S=0,\ \ T+U-V+S=0, \end{aligned}\right\} \ldots(45).$$

It thus appears that the four pentahedra have three faces, and consequently one synhedral point, in common; that these faces are the diagrammatic planes of the octahedron; that the triangular faces of the pentahedra are the faces of the octahedron; and that their synhedral lines are in one plane (S). The entire system thus consists of six angular points, fifteen edges, and eleven faces; it has some analogy to the complete tetragon, and may be denominated a *complete hendecahedral* sexangle.*

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* The adjective 'hendecahedral' is added to distinguish this system from the complete sexangle *in its general state* which has *twenty* faces.

The equation to the surface of the second degree touching the edges of the diagrammatic octahedron, of course is

$$T^2 + U^2 + V^2 = \tfrac{1}{2}S^2 \ldots\ldots\ldots (46);$$

also the equations to the surfaces of the second degree touching the edges of the sexangular pentahedra, are

$$\left.\begin{aligned} S^2 &= 4UV + 4VT + 4TU \\ S^2 &= 4UV - 4VT - 4TU \\ S^2 &= -4UV + 4VT - 4TU \\ S^2 &= -4UV - 4VT + 4TU \end{aligned}\right\} \ldots\ldots\ldots (47).$$

Equation (13) is of course the general equation to surfaces of the second degree circumscribed about the hendecahedral sexangle, and (9) the general equation to surfaces of the second degree inscribed in the diagrammatic octahedron. It is easily seen that such of the surfaces (46) and (47) as touch the same edge have a common point of contact. The equations to the points of contact of the various edges will be found to be as follows:

$$\left.\begin{aligned} aa', &\quad U = V = S = 0, \\ bb', &\quad V = T = S = 0, \\ cc', &\quad T = U = S = 0, \\ bc, &\quad T = 0, \quad U = V = -\tfrac{1}{2}S, \\ b'c', &\quad T = 0, \quad U = V = \tfrac{1}{2}S, \\ ca, &\quad U = 0, \quad V = T = -\tfrac{1}{2}S, \\ c'a', &\quad U = 0, \quad V = T = \tfrac{1}{2}S, \\ ab, &\quad V = 0, \quad T = U = -\tfrac{1}{2}S, \\ a'b', &\quad V = 0, \quad T = U = \tfrac{1}{2}S, \\ ab', &\quad V = 0, \quad -T = U = \tfrac{1}{2}S, \\ a'b, &\quad V = 0, \quad T = -U = \tfrac{1}{2}S, \\ ac', &\quad U = 0, \quad -T = V = \tfrac{1}{2}S, \\ a'c, &\quad U = 0, \quad T = -V = \tfrac{1}{2}S, \\ bc', &\quad T = 0, \quad -U = V = \tfrac{1}{2}S, \\ b'c, &\quad T = 0, \quad U = -V = \tfrac{1}{2}S, \end{aligned}\right\} \ldots\ldots (48)$$

Many other relations of the system might be given, and most of the preceding theorems that have reference to octahedra and pentahedra might be enunciated as properties of the complete hendecahedral sexangle, but I hasten to conclude, and shall therefore merely insert the following theorem.

XVII. *In a complete hendecahedral sexangle the synhedral point common to the pentahedra is the pole of the plane of their four*

synhedral lines relative to any of the following surfaces of the second degree—the surface touching the edges of any of the five solid figures, any surface circumscribed about the system, and any surface inscribed in the octahedron. If any surface of the second degree be circumscribed about the system, the tangent planes drawn at corresponding angles will intersect in three straight lines in the plane of the synhedral lines. Such of the five surfaces touching the edges of the five solid figures of the system, as touch the same edge, have a common point of contact on that edge; and, omitting the three edges common to the pentahedra, the six straight lines joining the points of contact of corresponding edges intersect in the synhedral point common to the pentahedra.

To the properties given in this paper must be added the first ten theorems of my first memoir on the "Theorems in Space analogous to those of Pascal and Brianchon in a Plane," (*Journal*, vol. IV., New Series, pp. 26–38), remembering in the latter to read 'diagrammatic' instead of 'diagonal' plane, and in the equations that relate to the octahedron to write $\frac{1}{2}S$ for S.

York Town, near Bagshot,
Jan. 26th, 1852.

ON THE CALCULUS OF DISTANCES, AREAS, AND VOLUMES, AND ITS RELATIONS TO THE OTHER FORMS OF SPACE.

By Thomas Cotterill.

Suppose we take any number of points in a plane, and through them draw all the possible right lines, which will again intersect in a certain number of points, which in reference to the original we may call derived points. The geometry of this figure would be known when we had discovered all the relations between the distances of these points, the angles between the lines, and the areas of the triangles. The investigation of the two first classes seems to form the object of Trigonometry. But the relations of the areas, even of the original points, when their number is greater than four, has not (that I am aware of) been examined; and yet, besides its own intrinsic interest, when we consider the importance, both in the Geometry of Position and the Theory of Curves, of certain limiting cases, the subject is evidently one of considerable interest.

On taking the figure thus formed from six points, and endeavouring to express the area of a Pascal triangle in terms of the original areas, I was fortunate enough to hit upon a method by which this could be done with consider-

able facility and equally applicable to all such cases. The result was, that that peculiar function of the six points, which we may call the conic, could be expressed in fifteen binomials each containing the original areas, and in sixty monomials each involving a Pascal triangle as a factor, in such a way that if one of the original points moved so that a Pascal triangle was constant in area, the point described a conic section.

Further examination shewed that the means and instruments by which this was done were equally effective in discovering the relations arising from introducing a fresh point or line; and that besides, by different operations, they could be employed with almost equal advantage to obtain trigonometrical properties. The expressions can also be dualized, either independently or by the method of polar reciprocals.

The question then arose, Can the method be applied to general space? The answer is, Every relation in space gives relations in all other spaces, but sometimes in forms which cannot be called analogues. But there are certain functions whose forms are the same in each space, but differ in their number of terms, and sometimes in the value of a constant.

Amongst these, by far the most important is one which in a different shape is well known, but does not appear to have met with the attention it deserves in the form in which it will here be put.

The explanation of the rule of signs of areas and volumes is given by Mœbius in his *Lehrbuch der Statik*, but admits of simplification. The signs of the trigonometrical functions are made to depend on those already given, and are perfectly free from ambiguity.

Taking space of three dimensions and using a notation which will be easily understood, we have the equation of addition of volumes

$$V = P\alpha + P\beta + P\gamma + P\delta,$$

which in a slightly different form has a pretty enough geometry of its own, instances of which in space of two dimensions are given by Mr. Moon in vol. v. p. 131 of this Journal, and can easily be increased.

I am not aware however that it has ever been observed (though it is strange if it has not), that by substituting the expressions for the volumes in terms of the altitudes and bases, and supposing the variable point to be in one of the planes, we obtain the equation to the plane, and also the

perpendicular on this plane from any point in a form which explains the origin of the double sign in the common expression. Again, taking another point, subtracting and substituting trigonometrically, we obtain an equation which gives the properties of the tetrahedron.

Now take the equation $V = \Sigma P\alpha$, and the theorem that triangles and tetrahedrons of the same altitude are to one another as their bases. From these two properties we deduce the equation

$$V.P\tau = \Sigma P\alpha \,.\, A\tau,$$

where $P\tau$ is either the shortest distance between the point P and the plane τ, or the tetrahedral volume subtended at P by the triangle τ. It is scarcely possible to overrate the importance of this equation, as I believe that there is no general geometrical relation which cannot be proved from it, and one simple definition founded on the principle of similarity.

This expression is endowed with complete duality. In space of two dimensions, the limiting case (when τ is any line through P) is discussed by Mr. Salmon in the 1st chapter of his *Higher Plane Curves.**

My present object is to shew its use in explaining a point in the theory of curves and curved surfaces considered as envelopes, which requires some explanation. I shall give the formulæ in space of three dimensions, though it must be remembered that they may be increased in space of two dimensions. But first I must explain one point: suppose by means of the foregoing formula we have transformed any expression homogeneous or not involving the coordinates of a point with regard to any number of planes, into another involving the coordinates of the same point with regard to other planes. Let the line ps through the point p meet the plane σ in s; then if $p\sigma$ be the ordinate of p, $p\sigma = ps$, $\sin(ps, \sigma)$. For each ordinate substitute a similar expression, and take p at infinity on the line ps; the terms involving the highest powers of the coordinates will alone remain with sines of angles substituted for ordinates. We have in fact obtained *a porismatic asymptotic hyperdeterminant.*

Let, as before, Greek letters denote planes: then $3V\cos\pi\tau = \Sigma\alpha A_\tau \cos\alpha\pi$ and $3V\cos\alpha\pi = \alpha A_\pi + \beta B_\pi \cos\beta\alpha + \gamma C_\pi \cos\gamma\alpha + \delta D_\pi \cos\delta\alpha$. Substituting in the same way for β, we have

$$gV^2 \cos\pi\tau = \Sigma\alpha^2 A_\pi A_\tau + \alpha\beta\,(A_\pi B_\tau + A_\tau B_\pi)\cos\alpha\beta\,;$$

therefore $\quad = -\Sigma(A_\pi - B_\pi)(A_\tau - B_\tau)\,\alpha\beta\cos\alpha\beta.$

* It is impossible to mention this work, whilst fresh from its perusal, without expressing the pleasure and instruction received.

Hence $gV^2 = -\Sigma(A_\pi - B^\pi)^2\,\alpha\beta\cos\alpha\beta = \Omega$ suppose ;*

therefore $$2\Omega\cos\pi\tau = \Sigma A_\pi \frac{d\Omega}{dA_\tau}.$$

Now suppose we have an expression involving tangential coordinates, and have transformed it into another expression by means of a formula such as $VP_\tau = \Sigma P\alpha A_\tau$. Then if n be the dimensions of the expression which we had better suppose homogeneous, the transformed expression will consist of terms such as $\Omega^r u_{n-2r}$; u_{n-2r} being a homogeneous function of the coordinates of the degree $n-2r$. Now equations of either point or tangent coordinates are porismatized (using a word and idea of Mr. Gompertz, in his ingenious work on Imaginary Quantities), by substituting for instance for A_τ, $A_\pi + \lambda A_\tau$, and equating coefficients of like powers of λ. Taking the coefficients of the first power of λ, and substituting, we shall at last obtain an equation involving n porismatic points or planes. Let us see the effect of one such substitution on the term $\Omega^r u_{n-2r}$. The coefficient of λ evidently is

$$\Omega^r \Sigma D_\pi \frac{dU_{n-2r}}{dD^\tau} + 2r\,U_{n-2r}\Omega^{r-1}\Sigma A_\pi \frac{d\Omega}{dA_\tau}$$

$$= \Omega^r\left(\Sigma D_\pi \frac{dU_{n-2r}}{dD_\tau} + 2r\cos\pi\tau\, U_{n-2r}\right).$$

Thus in space of two dimensions the polar of $A_tB_t = b^2$ is $A_tB_s + B_tA_s = 2b^2\cos_s t$.

But taking again our tangential equation, by supposing the plane to be at infinity, we must have the coefficients of the highest dimensions of the coordinates to be equal. Now in Plucker's *Canonical Form*, the highest dimension consists of one term. If then infinity is a tangent, and therefore the sum of the coefficients nothing, this term would seem to disappear. But remember that the perpendicular from an infinite point in a given direction (in fact its ordinate) with respect to a given plane, varies as the sine of the angle between the two.

One example is the parabola; another is the curve $\Sigma A_t(B_t - C_t)^2\cot A = 0$, which, by means of the equation $2K = \Sigma(B_t - C_t)^2\cot A$, can be put under the form

$$A_t = -2\rho\sin ct\sin bt,\cos at.$$

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* Compare Salmon, p. 11, and the beautiful explanation there given of $\Omega = 0$ in space of two dimensions.

Here sin $at = \frac{B_t - C_t}{a}$; and we see that the sum of the coefficients of the highest dimensions is nothing.

This is a very remarkable curve. Suppose we call the circle bisecting the six distances joining the three points *ABC*, and the intersection of the perpendiculars the bisecting circle. The intercepted portion of any tangent to the curve is equal to the diameter of the circumscribing circle, and its middle point is on the bisecting circle: the tangents at its extremities are at right angles to one another, and also meet in the bisecting circle and cut off equal segments of the sides. The sides of this right-angled triangle are as the curvatures at the points at which they touch. The curve is of the fourth order, and is that touched by the asymptotes of the rectangular hyperbolas passing through the points *ABC*.

Writing from recollection, I stated in the February No. of this Journal, that the tangents at the cusps met in the intersection of the perpendiculars. This is incorrect.

I have not alluded to the relations of distances on the same line. They are nothing but algebra, and the reader can form one of the functions employed by taking any of the expressions which are known of four points in harmonic proportion, and seeing how they can be deduced from each other. When he has thus formed an equation of perhaps sixteen sides, let him see that he has been working nothing but algebra, and that a great portion of the expressions can be derived from a single porismatic one. And so for the anharmonic and involution functions.*

49, *Coleshill Street, April 3rd*, 1852.

ON A PROBLEM IN COMBINATIONS.

By R. R. Anstice.

The object of this paper is to prove the following proposition. If n is odd and $6n + 1$ is a prime number, $12n + 3$ symbols can be arranged into $6n + 1$ series of triads, $4n + 1$ triads in each, each containing all the symbols; and in such a manner that every possible duad shall occur once. and once only, throughout the whole. Also all the series can be deduced

* Since the date of this paper, M. Chasles's invaluable work on Geometry has appeared. The functions here alluded to in their cases of ease form the basis of his system, and are made to depend on each other by the geometrical properties of the anharmonic ratio. I think, however, the knowledge that the expressions employed are only algebraic transformations of the same form, obtained for the most part by given rules, will be found to throw great light on these remarkable functions.

from any one of them by help of three cycles; one of a single term, the other two of $6n+1$ terms each.

As my reasoning will be based on Mr. Kirkman's conclusions, I would ask the reader to have at hand for reference two papers by that author in the Journal (new series, No. x. p. 191, and No. xxiv. p. 256).

Mr. Kirkman has shown in his first paper, that the number of symbols, all the duads of which can be exhibited but not repeated in a system of triads, must be of form $6n+1$ or $6n+3$. Of these two forms, $6n+3$ being divisible by 3, is the only one of course available for the purpose in view.

Next he shows that if this be true, not only for a number N of symbols, but also for $\frac{1}{2}(N-1)$, then the triplets of N may be exhibited in terms of the triplets of $\frac{1}{2}(N-1)$ and the duads of $\frac{1}{2}(N+1)$ symbols. It is to this case that I confine myself in the present paper.

Now N being of form $6n+3$, $\frac{1}{2}(N-1)$ will be of form $3n+1$, which number of symbols cannot be arranged into triads exhausting all the duads unless n is even. Therefore our number must be of form $12n+3$.

Here, when $n=1$ or the symbols are 15, the arrangement has been already made; but not I believe by cyclic changes. I will give the simplest rule I can for this. Put the symbols into 5 triads in any way you like for a primary arrangement. Then to form the first cycle take in succession one term from the first triad, one form the second, one from the third, one from the fourth, one from the fifth, and the remaining two from the fourth. To form the second cycle, take in succession one from the first triad, one from the third, one from the fifth, one from the second, another from the second, another from the fifth, another from the third. There still remains one term of the first cycle, which will form the cycle of one, or will remain unchanged. Thus for a primary arrangement take

$$| a_1a_2a_3 | b_1b_2b_3 | c_1c_2c_3 | d_1d_2d_3 | e_1e_2e_3 |$$

And we may take for a first cycle $a_1b_1c_1d_1e_1d_2d_3$, and for a second $a_2c_2e_2b_2b_3e_3c_3$, while a_3 remains unchanged; and by help of these cycles form all the arrangements as follows:

$$\left|\begin{array}{ccccc}
a_1a_2a_3 & b_1b_2b_3 & c_1c_2c_3 & d_1d_2d_3 & e_1e_2e_3 \\
b_1c_2a_3 & c_1b_3e_3 & d_1e_2a_2 & e_1d_3a_1 & d_2b_2c_3 \\
c_1e_2a_3 & d_1e_3c_3 & e_1b_2c_2 & d_2a_1b_1 & d_3b_3a_2 \\
d_1b_2a_3 & e_1c_3a_2 & d_2b_3e_2 & d_3b_1c_1 & a_1e_3c_2 \\
e_1b_3a_3 & d_2a_1c_2 & d_3e_2b_2 & a_1c_1d_1 & b_1c_3e_2 \\
d_2e_3a_3 & d_3c_2e_2 & a_1c_3b_3 & b_1d_1e_1 & c_1a_2b_2 \\
d_3c_3a_3 & a_1e_2b_2 & b_1a_2e_3 & c_1e_1d_2 & d_1c_2b_3
\end{array}\right|$$

Now to prove that these cycles will be efficient, I observe that, by the principle of cyclic changes, the third arrangement is derived from the second by the same operation as the second from the first: the fourth from the second by the same operation as the third from the first, and so on. Therefore, if we can prove that no duad of the *first* arrangement can recur until the cycle is repeated, we prove at the same time that no duad of *any* of the arrangements will recur. But in the first arrangement the duads agree in letter and only differ in subscript. Suppressing the subscripts, our two cycles become *abcdedd* and *acebbec*.

Now write down either of these cycles, and underneath it write either the same cycle or the other, beginning at any term; and we shall find the letters agree in never more than one place: of course excluding the case where the same identical cycle is written underneath, beginning at the same term. A little thought will show that this proves their efficiency. Or we may do it mechanically thus. Take two circular discs of equal diameter and at equal distances round the circumference of the one and the other respectively: write the two cycles. Do the same in the case of two other equal circular discs of different diameter to the former. Apply two of the discs of unequal diameter concentrically to each other; and in any position we shall find them agree in letter in never more than one place, excepting only when two identical cycles are made to agree in *every* place.

The rule, that two cycles should be efficient *in combination* seems difficult to give. But a simple rule may be given, that a cycle shall be efficient *alone*. For suppose the cycle inefficient, then the interval between two letters, b and c suppose, must be the same as that between another b and another c measured in the same direction. From which it follows, that the interval between the b's and c's themselves must be the same. Consequently, that a cycle may be efficient by itself, the intervals between like letters (neglecting subscripts) must produce a series of numbers all different. But (between the two b's suppose) there are *two* intervals whose sum is the number of terms of the cycle. Thus that a cycle may be efficient alone, we have twice as many conditions as there are repetitions of letter in the cycle.

Now look at the second of the cycles. The letters to be repeated are written in a certain order, then in the reverse order. The intervals between like letters, then, are the odd numbers 1 3 5. The complements of the same are 6 4 2; all different numbers.

Generally consider a cycle of $2n+1$ terms, consisting of one unrepeated and n repeated letters. Write the unrepeated letter: then the letters which are to be repeated, in any order, and then in the reverse order, and you have a cycle efficient not to reproduce duads, for the intervals between like letters will be all the odd numbers less than $2n+1$; and the complements of these numbers will be the even numbers less than $2n+1$. This cycle is useful to arrange the duads which can be formed of $2n$ symbols into $2n-1$ series, of n duads in each; each series containing all the symbols.

Write n duads containing all the symbols to begin with, and for the cycle take one term in succession from each of the duads to the nth, and then take a term from these duads in the reverse order till we return to the second duad. There still remains one term of the first duad for the cycle of one term.

Thus for 8 symbols, take to begin with the four duads

$$| a_1a_2 | b_1b_2 | c_1c_2 | d_1d_2 |$$

and our cycle may be $a_1b_1c_1d_1d_2c_2b_2$, while a_2 remains unchanged.

For 12 symbols, take to begin with the 6 duads

$$| a_1a_2 | b_1b_2 | c_1c_2 | d_1d_2 | e_1e_2 | f_1f_2 |$$

and our cycle may be $a_1b_1c_1d_1e_1f_1f_2e_2d_2c_2b_2$, while a_2 remains unchanged. These arrangements are subsidiary, as Mr. Kirkman has shown, to the problem we have in view. It will be more convenient, however, to express the primary arrangement in terms of the successive members of the cycle which will produce the others. Thus, in the case of 8 symbols, let k be the constant term, and $a_0a_1a_2a_3a_4a_5a_6$ the successive members of the cycle of 7. Our primary arrangement, expressed in terms of these, will be $ka_0 \mid a_1a_6 \mid a_2a_5 \mid a_3a_4 \mid$; and by successively increasing the subscripts by 1, throwing out the sevens, we shall complete a system comprising all the duads, and every line thereof comprising all the symbols.

But in order to achieve our problem it is further necessary, as Mr. Kirkman has shown, to construct another similar system, the four duads in any given line of which shall be taken from a different both line and column of the former system. To get the primary arrangement of this new system, take for the first duad ka_0 as before, for the second take a_1a_3, a_3 being associated with a_1, because 3 is a primitive root of 7, and form the other duads from this by continually multiplying the subscripts by the square of 3 to modulus 7, that is by 2. We thus get the primary arrangement of the new system

$ka_0 \mid a_1a_3 \mid a_2a_6 \mid a_4a_5 \mid$; and using the same cycle for both, we get the two systems

ka_0	a_1a_6	a_2a_5	a_3a_4
ka_1	a_2a_0	a_3a_6	a_4a_5
ka_2	a_3a_1	a_4a_0	a_5a_6
ka_3	a_4a_2	a_5a_1	a_6a_0
ka_4	a_5a_3	a_6a_2	a_0a_1
ka_5	a_6a_4	a_0a_3	a_1a_2
ka_6	a_0a_5	a_1a_4	a_2a_3

ka_0	a_1a_3	a_2a_6	a_4a_5
ka_1	a_2a_4	a_3a_0	a_5a_6
ka_2	a_3a_5	a_4a_1	a_6a_0
ka_3	a_4a_6	a_5a_2	a_0a_1
ka_4	a_5a_0	a_6a_3	a_1a_2
ka_5	a_6a_1	a_0a_4	a_2a_3
ka_6	a_0a_2	a_1a_5	a_3a_4

which will be found to fulfil the conditions proposed. This twofold arrangement of the duads of 8 symbols has been given by Mr. Kirkman (No. XXIV. p. 261), but he thinks that it would be difficult to give for higher values of $2n$ than 8. However, it may always be done when $2n - 1$ is prime and n is even: thus in the case of 12 symbols, let k be the constant term $a_0a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}$, the successive members of the cycle of eleven terms. The primary series of our first system, expressed in terms of these, will be

$$ka_0 \mid a_1a_{10} \mid a_2a_9 \mid a_3a_8 \mid a_4a_7 \mid a_5a_6 \mid .$$

The primary series of our second system, similarly expressed, may be

$$ka_0 \mid a_1a_2 \mid a_4a_8 \mid a_5a_{10} \mid a_9a_7 \mid a_3a_6 \mid .$$

Here, with k is associated a_0 as before. With a_1 is associated a_2, 2 being a primitive root of 11. And from this as base the other duads are derived by successively multiplying the subscripts by the square of two, or four, throwing out elevens. If the same cycle is used to complete the two systems, they will be found on trial, and may readily be proved, to fulfil the conditions proposed. The proof will be involved in that of the proposition which I have proposed to establish.

But to return to the problem of 15: let k be the constant term

$$\left.\begin{matrix} p_0p_1p_2p_3p_4p_5p_6 \\ q_0q_1q_2q_3q_4q_5q_6 \end{matrix}\right\} \begin{matrix} \text{the successive members of} \\ \text{the two cycles.} \end{matrix}$$

Then the primary arrangement, expressed in terms of these, will be

$$\mid kp_0q_0 \mid p_1q_3q_4 \mid p_2q_1q_6 \mid p_3p_5p_6 \mid p_4q_2q_5 \mid .$$

Here the first triad consists of the constant term associated with the first member of each cycle. Then there are three triads (the 2nd, 3rd, and 5th), where one term of the first cycle is associated with a duad of the second. The formation of the duads is obvious; the sum of their subscripts equals 7.

To find the subscript of the associated term of the first cycle, multiply the square of the product of the subscripts of the duads by two, and throw out the sevens, or multiply the fourth power of either of the subscripts by two, and throw out the sevens. The remaining members of the first cycle form the fourth triad. We may complete all the arrangements from this primary one by successively increasing the subscripts by unit, throwing out the sevens.

To prove the efficiency of these cycles, it is sufficient, as I have said, to show that no duad in the primary arrangement can recur till the cycle is repeated. These duads may be arranged into three classes.

The first comprising duads whose terms are taken from the first cycle only; the second comprising those whose terms are taken from the second cycle only; the third comprising those whose terms are taken each from a different cycle. Now the cyclic changes which from any given duad of the first arrangement produce the duads corresponding to it in the others, will affect neither *its class* nor the *difference of its subscripts estimated to modulus* 7.

Therefore it will be sufficient to prove that in the primary arrangement these differences, in each of the 3 classes, produce a series of different numbers. Moreover, in the two first classes these differences must be estimated both positively and negatively to modulus 7, or, which comes to the same thing since 7 is prime, the *square* of the differences may be taken. In the third class we must estimate the differences always in the same manner, always subtracting, suppose, the subscript of the q from the subscript of the p.

The duads of the first class all occur in the fourth triad of the primary arrangement. The squares of the differences of their subscripts are (to modulus 7) 4, 2, and 1, all different numbers.

The duads of the second class occur in the 2nd, 3rd, and 5th triad of the primary arrangement. The squares of the differences of their subscripts (to modulus 7) are 1, 4, and 2. All different numbers.

The duads of the third class occur in the 1st, 2nd, 3rd, and 5th triad of the primary arrangement. The differences of their subscripts estimated as defined above, are 0, 5, and 4, 1 and 3, 2 and 6. All different numbers.

There is still a fourth class of duads, where the constant term is associated with a term from one or other of the cycles. These manifestly cannot recur till the cycle is repeated. Thus the efficiency of the cycles is demonstrated. I may

remark that these cycles will produce combinations of triads of Mr. Kirkman's form. (See *Journal* No. XXIV. p. 260, or the arrangement already given.) Indeed, it was from studying this form that I discovered them. But it is a mistake to suppose that all the combinations of triads which can be formed belong to one form. There is another pair of cycles, equally efficient with that already given, which will give combinations of triads of an entirely distinct form. The primary arrangement, expressed in terms of the successive members of these cycles, will be

$$kp_0q_0 \mid p_2q_1q_3 \mid p_4q_2q_6 \mid p_1q_4q_5 \mid p_3p_5p_6.$$

The first triad is as before. In the three next a duad of the second cycle is associated with a single term of the first. To find the duads proceed as follows: with q_1 join q_3, 3 being a primitive root of 7, and from this as base form the other duads by multiplying the subscripts by the square of 3 to modulus 7, that is by 2.

The subscript of the associated term of the first cycle is the semisum (estimated to modulus 7) of the subscripts of the duad. The three remaining members of the first cycle form the last triad. The efficiency of the cycles may be proved by estimating the differences of the subscripts as before.

Now I say that the combination of triads effected by these cycles will be entirely distinct from the former. To show this, use for a moment the former notation. Write the primary arrangement (in a slightly different order), and over each triad write its characteristic letter, thus:

$$\begin{array}{ccccc} a & b & c & d & e \\ \mid kp_0q_0 \mid & p_1q_4q_5 \mid & p_2q_1q_3 \mid & p_3p_5p_6 \mid & p_4q_2q_6 \mid \end{array}$$

and we have (according to our former notation) the two cycles *abcdedd* and *acecbbe;* where any of the three subscripts 1, 2, 3 may be appended to any of the letters, provided we do not use the same subscript twice to the same letter. Thus we may take for cycles $\begin{matrix} a_1b_1c_1d_1e_1d_2d_3 \\ a_2c_2e_2c_3b_3b_3e_3 \end{matrix}$,

the first of which is the same as before, the other different. Using these cycles to complete the system, we get

$$\begin{array}{|ccc|ccc|ccc|ccc|ccc|} a_1a_2a_3 & & & b_1b_2b_3 & & & c_1c_2c_3 & & & d_1d_2d_3 & & & e_1e_2e_3 & & \\ b_1c_2a_3 & & & c_1b_3e_3 & & & d_1e_2b_2 & & & e_1d_3a_1 & & & d_2c_3a_2 & & \\ c_1e_2a_3 & & & d_1e_3a_2 & & & e_1c_3b_3 & & & d_2a_1b_1 & & & d_3b_2c_2 & & \\ d_1c_3a_3 & & & e_1a_2c_2 & & & d_2b_2e_3 & & & d_2b_1c_1 & & & a_1b_3e_2 & & \\ e_1b_2a_3 & & & d_2c_2e_2 & & & d_3b_3a_2 & & & a_1c_1d_1 & & & b_1e_3c_2 & & \\ d_2b_3a_3 & & & d_3e_2c_3 & & & a_1e_3c_2 & & & b_1d_1e_1 & & & c_1a_2b_2 & & \\ d_3e_3a_3 & & & a_1c_3b_3 & & & b_1a_2e_2 & & & c_1e_1d_2 & & & d_1c_2b_3 & & \end{array}$$

a combination of triads entirely distinct from the other in form.

Now I say that whenever n is odd and $6n + 1$ prime, $12n + 3$ symbols can be exhibited in $6n + 1$ series of triads, each series containing all the symbols, and the whole exhausting but not repeating all the duads, by help of a pair of cycles similar to the second of those I have explained in the case of 15.

Let k be the constant term,

$$\left.\begin{matrix} P_0, P_1, P_2 \ldots\ldots P_{6n-2}, P_{6n-1}, P_{6n} \\ Q_0, Q_1, Q_2 \ldots\ldots Q_{6n-2}, Q_{6n-1}, Q_{6n} \end{matrix}\right\}$$

be the successive members of the two cycles. Let r be a primitive root of $6n + 1$. Let ρ be any value of x which satisfies the equivalence

$$\frac{x^{3n} + 1}{x + 1} \equiv 0, \text{ (modulus } 6n + 1).$$

That is to say, let

$$\rho \equiv r^{\mu}, \text{ (modulus } 6n + 1),$$

where μ is any odd number not divisible by $3n$.

Then the primary arrangement, expressed in terms of these will consist,

(1). Of a triad in which the constant term is associated with one member from each of the cycles.

This will in every case be kP_0, Q_0.

(2). Of $3n$ triads, in each of which one term of the first cycle is associated with a duad of the second. These triads will be of the form

$$\Sigma P r^{2i} \underline{\frac{\rho + 1}{2}}\ Q r^{2i} Q r^{2i} \rho,$$

where all integral values are to be substituted for i from 0 to $3n - 1$.

(3). Of n triads, each consisting of terms from the first cycle only. These triads will be of the form

$$\Sigma P r^{2i+1} \underline{\frac{(\rho+1)}{2}}\ P r^{2i+2n+1} \underline{\frac{(\rho+1)}{2}}\ P r^{2i+4n+1} \underline{\frac{(\rho+1)}{2}}$$

where all integral values are to be substituted for i from 0 to $n - 1$. The subscripts are of course all estimated to modulus $6n + 1$.

For, if possible, let the cycles be inefficient.

First, if possible, let the difference of the subscripts of two duads of the second class be equivalent, estimated positively or negatively to modulus $6n+1$; that is, if possible, in the two triads

$$P r^{2i}_{\frac{\rho+1}{2}} Q r^{2i} Q\rho r^{2i} \text{ and } P r^{2i}_{\frac{\rho+1}{2}} Q r^{2i} Q\rho r^{2i}.$$

Let $\rho r^{2i} - r^{2i} \equiv \pm(\rho r^{2i_1} - r^{2i_1})$, (modulus $6n+1$);

therefore, dividing by $\rho - 1$,

$$r^{2i} \equiv \pm r^{2i_1}, \text{ (modulus } 6n+1),$$

i and i_1 being by hypothesis different, one of them must be the greater. Let $i > i_1$; therefore dividing by r^{2i_1}, we must have $r^{2(i-i_1)} \equiv \pm 1$, (modulus $6n+1$).

To satisfy this equivalence, we must have, if we take the upper sign $2(i-i_1) = 6\lambda n$, λ being some integer, the value $\lambda = 0$ inadmissible because i and i_1 are different, and so is any other value, for $i > i_1$ and both are less than $3n$; therefore $i - i_1$ is positive and less than $3n$, and $2(i-i_1)$ less than $6n$.

If we take the lower sign, we must have

$$2(i-i_1) = 3n + 6\lambda n,$$

λ being some integer, that is to say, an even number equal to an odd one, which is absurd.

Next, if possible, let the difference of subscripts of the duads of the third class in these same triads be equivalent. But we have proved the difference of subscripts of the duads of the second class not to be equivalent. And the differences in question, being the halves of these former differences to modulus $6n+1$, must likewise not be equivalent.

Lastly, if possible, let the differences of the subscripts, estimated positively or negatively to modulus $6n+1$, of two duads of the first class be equivalent.

(1). Let the duads occur in the same triad,

$$P r^{2i+1}_{\frac{\rho+1}{2}} P r^{2i+2n+1}_{\frac{\rho+1}{2}} P r^{2i+4n+1}_{\frac{\rho+1}{2}};$$

therefore, (expunging the common factor $r^{2i+1}_{\frac{\rho+1}{2}}$), we must have two of the three quantities

$$r^{2n} - 1, \quad r^{4n} - 1, \quad r^{4n} - r^{2n}$$

equivalent, estimated positively or negatively.

$$\left.\begin{array}{ll} \text{Let} & r^{4n} - 1 \equiv \pm (r^{2n} - 1); \\ \text{therefore either} & r^{2n} - 1 \equiv 0, \\ \text{or} & r^{2n} \equiv 0, \\ \text{or else} & r^{2n} \equiv -2, \\ \text{and therefore} & 2^3 + 1 \equiv 0. \end{array}\right\} \text{(modulus } 6n + 1\text{).}$$

All impossible. And so of the others.

(2). Let the duads occur in two different triads

$$P_{r^{\frac{2i+1}{2}}}^{\frac{\rho+1}{2}} \; P_{r^{\frac{2i+2n+1}{2}}}^{\frac{\rho+1}{2}} \; P_{r^{\frac{2i+4n+1}{2}}}^{\frac{\rho+1}{2}} \text{ and } P_{r^{\frac{2i_1+1}{2}}}^{\frac{\rho+1}{2}} \; P_{r^{\frac{2i_1+2n+1}{2}}}^{\frac{\rho+1}{2}} \; P_{r^{\frac{2i_1+4n+1}{2}}}^{\frac{\rho+1}{2}}.$$

If we take corresponding duads from the two triads, the difference of subscripts are obviously not equivalent, estimated positively or negatively, from the same reasoning as before. Take then two duads which do not correspond, say the first and third term of the one triad and the first and second of the other; therefore we must have (expunging the common factor $\frac{r(\rho+1)}{2}$),

$$\left.\begin{array}{ll} & r^{2i}(r^{4n} - 1) \equiv \pm r^{2i_1}(r^{2n} - 1); \\ \text{therefore} & r^{2(i-i_1)} \times (r^{2n} + 1) \equiv \pm 1, \\ \text{or, since} & 1 + r^{2n} + r^{4n} \equiv 0, \\ & r^{4n+2(i-i_1)} \equiv \mp 1. \end{array}\right\} \text{(modulus } 6n + 1\text{).}$$

To satisfy this equivalence, if we take the upper sign, we may reduce, as before, to the absurdity of an even number equalling an odd one. If we take the lower, we must have

$$\left.\begin{array}{rl} & 4n + 2(i - i_1) = 6\lambda n \\ \text{or} & 2n + i - i_1 = 3\lambda n \end{array}\right\} \lambda \text{ being some integer.}$$

But both i and i_1 lie between the limits (inclusive) 0 and $n - 1$, and therefore $i - i_1$ between the limits $\pm (n - 1)$, and $2n + i - i_1$ between the limits $n + 1$, $3n - 1$.

Therefore the above equation is impossible, and similarly of the other duads which do not correspond.

There remain to be considered the three duads which occur in the triad kP_0Q_0 of the primary arrangement.

Two of them are duads of the fourth class and cannot recur till the cycle is repeated. The other is of the third class, and the difference of its subscripts = 0: while the difference of subscripts of no other duad of the third class can be equivalent to 0. Thus the efficiency of the cycles not to reproduce duads is demonstrated. Moreover, the primary

arrangement (and therefore all the arrangements) will contain all the symbols. This follows from two well-known propositions in the theory of numbers.

PROP. I. If $2N+1$ is a prime number, and r a primitive root thereto, not only will the different terms of the series

$$r^0\ r^1\ r^2 \ldots\ldots r^{2N-2}\ r^{2N-1},$$

estimated to modulus $2N+1$, comprise all the natural numbers from 1 to $2N$; but these terms, all multiplied by any of the same number, will do so likewise, provided that number is itself prime to $2N+1$. From this it follows that to the letter P all the subscripts are annexed.

PROP. II. The same things supposed, let ρ be any value of x which satisfied the equivalence

$$x^N + 1 \equiv 0, \text{ (modulus } 2N+1).$$

Then the different terms of the two series

$$\left.\begin{array}{llll} r^0 & r^2 & r^4 \ldots\ldots & r^{2(n-2)}\ \ r^{2(n-1)} \\ \rho r^0 & \rho r^2 & \rho r^4 \ldots\ldots & \rho r^{2(n-2)}\ \ \rho r^{2(n-1)} \end{array}\right\}$$

will comprise all the natural numbers from 1 to $2N$.

From hence it follows that to the letter Q all the subscripts are annexed.

To apply our formula, let $n=3$. Then $6n+1=19$, and is prime. A primitive root of 19 is 10. Taking then $r=\rho=10$, we find at once, by help of Jacobi's *Canon. Arith.*, the primary arrangement of 39 symbols, as follows:

$$\begin{aligned} & kp_0q_0 \\ & + p_{15}q_1q_{10} + p_{18}q_6q_{12} + p_{14}q_9q_3 + p_{13}q_{11}q_{15} + p_5q_{17}q_{18} + p_2q_8q_{14} \\ & + p_{10}q_7q_{13} + p_{12}q_{16}q_8 + p_3q_4q_2 \\ & + p_{17}p_{16}p_5 + p_9p_4p_6 + p_7p_1p_{11}. \end{aligned}$$

And by successively increasing the subscripts by unit, all the nineteen arrangements may be completed.

We may also obtain combinations of triads analogous to Mr. Kirkman's form, by help of cycles similar to the first of those explained in the case of 15. Our symbols having the same signification as before, the general expression for the primary arrangement will be as follows. It will consist—

(1). Of a triad in which the constant term is associated with one number from each of the cycles. This will be, as before, kP_0Q_0.

(2). Of $3n$ triads, in each of which one term of the first cycle is associated with a duad of the second. These triads will be of the form

$$\Sigma P_{\frac{\rho+1}{\rho-1} i^{3n+1}} Q_i Q_{6n+1-i},$$

where all integral values are to be substituted for i from 1 to $3n$. Or, as it may be otherwise expressed,

$$\Sigma P_{\frac{\rho+1}{\rho-1} r^{2i}} Q r^{2i} Q r^{2i+3n},$$

where all integral values are to be substituted for i from 0 to $3n-1$.

(3). Of n triads, each consisting of terms from the first cycle only. These triads will be of the form

$$\Sigma P_{\frac{\rho+1}{\rho-1} r^{2i+1}} P_{\frac{\rho+1}{\rho-1} r^{2i+2n+1}} P_{\frac{\rho+1}{\rho-1} r^{2i+4n+1}},$$

where all integral values are to be substituted for i from 0 to $n-1$. The subscripts are of course all estimated to modulus $6n+1$.

The proof of this presents no difficulty and is left to the reader.

To apply this formula take, as before, $n=3$,

$$r = \rho = 10.$$

Then $$\frac{\rho+1}{\rho-1} \equiv r^{14} \equiv r^{-4}, \text{ (modulus 19)},$$

and we can easily construct the following expression for the primary arrangement of 39 symbols:

$$\begin{aligned} &k p_0 q_0 \\ &+ p_{16} q_{12} q_{18} + p_6 q_2 q_{17} + p_9 q_3 q_{16} + p_7 q_4 q_{15} + p_4 q_5 q_{14} + p_1 q_6 q_{13} \\ &+ p_{17} q_7 q_{12} + p_5 q_8 q_{11} + p_{11} q_9 q_{10} \\ &+ p_8 p_{12} p_{18} + p_2 p_3 p_{14} + p_{10} p_{15} p_{13}. \end{aligned}$$

And by successively increasing the subscripts by unit, throwing out nineteens, complete the whole 19 arrangements.

The efficiency of either of these cycles may be tested, in the way already explained, by estimating the difference of the subscripts.

When n is even, I can obtain no solution of the problem in terms of either cycles.

POSTSCRIPT.

The number of distinct species of combinations of triads comprised in each of the general expressions given, will be $\frac{1}{2}(3n-1)$. For it is evident that if we multiply all the subscripts of a primary arrangement by any the same number, provided that number is itself prime to $6n+1$, we shall not affect the system resulting therefrom, but only the order in which the different arrangements composing that system are generated by the cycles. It is also evident that, in both the general expressions for the primary arrangement, such multiplication will either have no effect upon its form, or will change ρ into $\frac{1}{\rho}$ to modulus $6n+1$, according as the multiplier is equivalent to an even or an odd power of a primitive root. Consequently the different values of ρ, $3n-1$ in number, may be arranged in $\frac{1}{2}(3n-1)$ pairs, the constituents of any pair being inverse to each other to modulus $6n+1$, and either substituted in the primary arrangement producing the same combination. The number of distinct species therefore will be $\frac{1}{2}(3n-1)$ in each of the general expressions, or $3n-1$ altogether. Thus for fifteen symbols there will be two distinct species of combinations; for thirty-nine, eight; and so on.

February 7, 1852.

There is, I find, still another expression which gives efficient cycles. In the general expression, the proof of which I have given at length, we may take for the subscript of the associated member of the first cycle the *sum* of the subscripts of the duads of the second, instead of the semisum as there given; *i.e.* in all the subscripts of terms of the first cycle we may write $\rho+1$ in place of $\frac{1}{2}(\rho+1)$ where it occurs, and the proof will still hold good.

The two general expressions I originally gave generate systems mutually correspondent, depending on the twofold arrangement of duads which I first explained. For the duads of the second class, which are associated with P_0 in one of the systems when expanded, will be the duads of the second class which occur in the primary arrangement of the other system. In this new system, however, the correspondent system will be of the same form as the original one. I have applied it to 15 symbols, and find the form to be a distinct species, though closely resembling Mr. Kirkman's.

The number of distinct species of combinations of triads of $12n+3$ symbols (n being odd and $6n+1$ prime) will be then $\frac{3}{2}(3n-1)$. Or at least these are all which I can discover.

May 3, 1852.

I find the three expressions I have already given are all included in another still more general.

Let ρ, ρ_1 be any two different roots of the equivalence $x^{3n}+1 \equiv 0$, (modulus $6n+1$). The other symbols as before.

Then primary arrangement $= kP_0Q_0$

$$+ \Sigma P_{\frac{\rho_1-\rho}{\rho_1-1}} r^{2i}\, Q r^{2i}\, Q\rho r^{2i}$$

from $i=0$ to $i=3n-1$

$$+ \Sigma P_{\frac{\rho_1-\rho}{\rho_1-1}} r^{2i+1}\, P_{\frac{\rho_1-\rho}{\rho_1-1}} r^{2i+2n+1}\, P_{\frac{\rho_1-\rho}{\rho_1-1}} r^{2i+4n+1}$$

from $i=0$ to $i=n-1$.

The proof of this will be exactly similar to the proof which I have given of a particular case thereof, except in the matter of duads of the third class; and that part of the proof will result immediately from the two self-evident propositions in the theory of numbers which I have quoted. In fact the differences of the subscripts of duads of the third class will be the different terms of the two series

$$\Sigma \frac{1-\rho}{\rho_1-1} r^{2i} \text{ and } \Sigma \frac{1-\rho}{\rho_1-1} \rho_1 r^{2i}$$

(i being taken from 0 to $3n-1$), and will therefore be a series of numbers none of which can be equivalent.

Also number of distinct species of combinations $= \frac{3n(3n-1)}{2}$.

This number would at first sight appear to be the number of permutations of $3n$ things, taken in pairs; but that number is reduced one half by observing that the multiplication of all the subscripts of a primary arrangement by any odd power of a primitive root will have the same effect on its form as the simultaneous change of ρ into $\frac{1}{\rho}$ and ρ_1 into $\frac{1}{\rho_1}$.

I may add that I can now solve the problem when n is even. But too much space has been already devoted to such a trifle.

Wiggington, near Tring,
June 4, 1852.

END OF VOL. VII.

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